NUMERICAL METHODS

Unit I : Finite differences – difference table – operators E,Δ and - Relations between these operations – Factorial notation – Expressing a given polynomial in factorial notation – Difference equation – Linear difference equations – Homogeneous linear difference equation with constant coefficients.

Unit II: Interpolation using finite differences – Newton – Gregory formula for forward interpolation – Divided differences – Properties – Newton's formula for unequal intervals - Lagrange's formula – Relation between ordinary differences and divided differences – inverse interpolation.

Unit III: Numerical differentiation and integration – General Quadrature formula for equidistant ordinates – Trapezoidal Rule – Simpson's one third rule – Simpson's three eight rule – Waddle's rule – Cote's method.

Unit IV: Numerical solution of ordinary differential equations of first and second orders – Piccards method. Eulers method and modified Euleis method – Taylor's series method – Milne's method – Runge – Kutta method of order 2 and 4 – Solution of algebraic and transcendent equations. Finding the initial approximate value of the root – Iteration method – Newton Raphson's method.

Unit V: Simultaneous linear algebraic equations – Different methods of obtaining the solution – The elimination method by Gauss – Jordan method – Grouts' method – Method of factorization .

Books:

Calculus of finite differences and Numerical Analysis, P.P. Gupta & G.S. Malik, Krishna PrakashamMardin, Mecrutt.

UNIT I

FINITE DIFFERENCES

1.1 Introduction

We introduce the idea of finite differences and associated concepts, which have important applications in numerical analysis.

For example, interpolation formulae are based on finite differences. Through finite differences, we study the relations that exist between the values that are assumed by the functions whenever the independent variables change by finite jumps.

1.2 Difference Operations

There are three difference operators namely forward, backward and central difference operators.

Forward Difference Operator

Consider the function y = f(x). Suppose we are given a table of values of the function at the points

$$x_0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \cdots, x_n = x_0 + nh\ .$$

Let $f(x_0) = y_0,\ f(x_1) = y_1,\ f(x_n) = y_n$.

We define

$$\Delta[f(x)] = f(x+h) - f(x)$$

Thus $\Delta y_0 = f(x_0 + h) - f(x) = f(x_1) - f(x_0) = y_1 - y_0$.

$\therefore \Delta y_0 = y_1 - y_0$.

Further, $x_0, x_1, \cdots, x_n$ are called arguments. The corresponding values of f(x) are called entries and h is called the interval of differencing.

Similarly, $\Delta y_1 = y_2 - y_1$

$$\vdots \quad \vdots \quad \vdots$$

$$\Delta y_{n-1} = y_n - y_{n-1}$$

Δ is called the forward difference operator and $\Delta y_0, \Delta y_1, \cdots, \Delta y_{n-1}$ are called the first forward differences of the function y = f(x).

The second order differences of the function are defined by

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$

$$\vdots \quad \vdots \quad \vdots$$

$$\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}.$$

Similarly, higher order differences can be defined. In general the nth order differences are defined by the equations

$$\Delta^n y_i = \Delta^{n-1} y_{i+1} - \Delta^{n-1} y_i.$$

These differences of the function y = f(x) can be systematically represented in the form of a table called forward difference table. We can construct the difference table for any number arguments and a sample difference table is given for six consecutive arguments.

x	**y = f(x)**	**Δ y**	**Δ^2 y**	**Δ^3 y**	**Δ^4 y**	**Δ^5 y**
x_0	y_0					
		Δy_0				
$x_1 = x_0 + h$	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\Delta^2 y_3$			
		Δy_4				
$x_5 = x_0 + 5h$	y_5					

In the above forward difference table y_0 is known as the first entry and $\Delta y_0, \Delta^2 y_0, \cdots, \Delta^5 y_0$ are called leading differences.

Note:

Since each higher order difference is defined in terms of the previous lower differences by continuous substitution of each higher order difference can be expressed in terms of the values of the function.

Thus

$$\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)$$

$$= y_2 - 2y_1 + y_0$$

$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$

$$= (y_3 - 2y_2 + y_1) - (y_2 - 2y_1 + y_0)$$

$$= y_3 - 3y_2 + 3y_1 - y_0$$

$$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$$

$$= (y_4 - 3y_3 + 3y_2 - y_1) - (y_3 - 3y_2 + 3y_1 - y_0)$$

$$= y_4 - 4y_3 + 6y_2 - 4y_1 + y_0.$$

We observed that the coefficients occurring in the right hand side are simply the binomial coefficients in $(1-x)^n$. Hence in general, we have

$$\Delta^n y_0 = y_n - n_{c_1} y_{n-1} + n_{c_2} y_{n-2} - \cdots + (-1)^n y_0.$$

Properties of the Operator Δ:

1. Δ is linear, that is $\Delta[a\, f(x) + b\, g(x)] = a\,\Delta[f(x)] + b\,\Delta[g(x)]$ where a and b are constants.

   Proof:

$$\Delta[a\, f(x) + b\, g(x)] = [a\, f(x+h) + b\, g(x+h)] - [a\, f(x) + b\, g(x)]$$
$$= a[f(x+h) - f(x)] + b[g(x+h) - g(x)]$$
$$\therefore \Delta[\mathrm{a\, f(x) + b\, g(x)}] = \mathrm{a}\,\Delta[\mathrm{f(x)}] + \mathrm{b}\,\Delta[\mathrm{g(x)}].$$

2. $\Delta^m \Delta^n [f(x)] = \Delta^{m+n} [f(x)]$.

   Proof:

$$\Delta^m \Delta^n [f(x)] = (\Delta\Delta\cdots m\ times)(\Delta\Delta\cdots n\, times)\, f(x)\mathrm{g}$$
$$= \Delta\Delta\cdots(m+n\, times)\, f(x)$$
$$\Delta^m \Delta^n [f(x)] = \Delta^{m+n} [f(x)].$$

3. $\Delta[f(x)\, g(x)] = f(x+h)[\Delta\, g(x)] + g(x)\Delta[f(x)]$.

   Proof:

$$\Delta[f(x)\, g(x)] = f(x+h)g(x+h) - f(x)g(x)$$
$$= [f(x+h)g(x+h) - f(x+h)g(x)] + [f(x+h)g(x) - f(x)g(x)]$$

$$= f(x+h)[g(x+h) - g(x)] + g(x)[f(x+h) - f(x)]$$

$$\Delta[f(x)\, g(x)] = f(x+h)[\Delta g(x)] + g(x)\Delta[f(x)].$$

4. $\Delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}.$

Proof:

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{f(x+h)g(x) - f(x)g(x) + f(x)g(x) - f(x)g(x+h)}{g(x)g(x+h)}$$

$$= \frac{g(x)[f(x+h) - f(x)] - f(x)[g(x+h) - g(x)]}{g(x)g(x+h)}$$

$$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta[f(x)] - f(x)\Delta[g(x)]}{g(x+h)g(x)}.$$

Backward Differences

Consider the function y = f(x). Suppose we are given a table of values of the function at the points

$$x_0,\ x_1 = x_0 + h,\ x_2 = x_0 + 2h,\ \cdots,\ x_n = x_0 + nh.$$

Let $f(x_0) = y_0,\ f(x_1) = y_1,\ f(x_2) = y_2,\ \cdots,\ f(x_n) = y_n$.

We define

$$\nabla[f(x)] = f(x) - f(x-h)$$

Thus $\nabla y_1 = y_1 - y_0$

$\nabla y_2 = y_2 - y_1$

$\vdots \quad \vdots \quad \vdots$

$\nabla y_n = y_n - y_{n-1}$.

∇ is called the backward difference operator and $\nabla y_1, \nabla y_2, \cdots, \nabla y_n$ are called the first order backward differences of the function y = f(x).

The second order differences of the function y = f(x) are defined by

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_2 = \nabla y_3 - \nabla y_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}.$$

Similarly, higher order differences can be defined. In general, the n[th] order differences are given by,

$$\nabla^n y_i = \nabla^{n-1} y_i - \nabla^{n-1} y_{i-1}.$$

These differences of the function y= f(x) can be systematically represented in the form of a table called backward differences table.

x	$y = f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
x_0	y_0					
		∇y_0				
$x_1 = x_0 + h$	y_1		$\nabla^2 y_0$			
		∇y_1		$\nabla^3 y_0$		
$x_2 = x_0 + 2h$	y_2		$\nabla^2 y_1$		$\nabla^4 y_0$	
		∇y_2		$\nabla^3 y_1$		$\nabla^5 y_0$
$x_3 = x_0 + 3h$	y_3		$\nabla^2 y_2$		$\nabla^4 y_1$	
		∇y_3		$\nabla^3 y_2$		
$x_4 = x_0 + 4h$	y_4		$\nabla^2 y_3$			
		∇y_4				
$x_5 = x_0 + 5h$	y_5					

As in the case of Δ we can prove that ∇ is also linear.

Remark:

The relation between the two difference operators is given by $\nabla[f(x+h)] = \Delta f(x)$.

For, $\nabla[f(x+h)] = f(x+h) - f(x) = \Delta f(x)$

Similarly,

$$\nabla^2[f(x+2h)] = \nabla[f(x+2h) - f(x+h)]$$

$$= \nabla f(x+2h) - \nabla f(x+h) \quad (\text{since } \nabla \text{ is linear})$$

$$= \Delta f(x+h) - \Delta f(x)$$

$$= \Delta[f(x+h) - f(x)]$$

$$\nabla^2 [f(x+2h)] = \Delta^2 f(x).$$

In general,

$$\nabla [f(x+nh)] = \Delta^2 f(x).$$

Hence from the forward difference table of the function f(x) we can obtain backward differences of all orders.

Central Difference Operator

Sometimes it is convenient to employ another system of differences known as central differences. We define central difference operator δ as

$$\delta f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right).$$

Thus if $f(x_i) = y_i$ then we have

$$\delta y_{\frac{1}{2}} = y_1 - y_0$$

$$\delta y_{\frac{3}{2}} = y_2 - y_1$$

$$\vdots \qquad \vdots$$

$$\delta y_{\frac{n-1}{2}} = y_n - y_{n-1}$$

Here the subscript of δy is the average of the subscripts of the two members of the difference. The higher order differences can define similar to forward and backward differences.

$$\delta^2 y_1 = \delta y_{\frac{3}{2}} - \delta y_{\frac{1}{2}}$$

$$\delta^2 y_2 = \delta y_{\frac{5}{2}} - \delta y_{\frac{3}{2}}$$

$$\delta^3 y_{\frac{3}{2}} = \delta^2 y_2 - \delta^2 y_1 \text{ etc.}$$

These differences of the function y = f(x) can be systematically represented in the form of a table called central difference table.

x	$y = f(x)$	δ	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{\frac{1}{2}}$			
$x_1 = x_0 + h$	y_1		$\delta^2 y_1$		
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$	
$x_2 = x_0 + 2h$	y_2		$\delta^2 y_2$		

$x_3 = x_0 + 3h$ $x_4 = x_0 + 4h$	y_3 y_4	$\delta y_{\frac{5}{2}}$ $\delta y_{\frac{7}{2}}$	$\delta^2 y_3$	$\delta^3 y_{\frac{5}{2}}$	$\delta^4 y_2$

Fundamental theorem for Finite Differences

Let f(x) be a polynomial of degree n. Then the n^{th} difference of f(x) is a constant and all higher order differences are zero.

1.3 Factorial Function (or) Factorial Notation

Definition

A product of the form x(x-h) (x-2h) … [x-(n-1) h] is called a factorial function and is denoted by $x^{(n)}$.

$$\therefore x^{(n)} = x(x-h)(x-2h)\cdots[x-(n-1)h].$$

Thus $x^{(1)} = x$, $x^{(2)} = x(x-1)$ and $x^{(3)} = x(x-1)(x-2)$.

We observed that $x^{(n)}$ is a polynomial of degree n with leading coefficient 1.

The following theorem shows that the formula for the first difference of $x^{(n)}$ is obtained by the simple rule of differentiation.

Theorem:

$\Delta x^{(n)} = nh\,x^{(n-1)}$. In particular when h = 1, $\Delta x^{(n)} = n\,x^{(n-1)}$.

Proof:

$$\Delta x^{(n)} = (x+h)^{(n)} - x^{(n)}$$

$$= (x+h)x(x-h)\cdots[x-(n-2)h] - x(x-h)(x-2h)\cdots[x-(n-1)h]$$

$$= x(x-h)(x-2h)\cdots[x-(n-2)h]\,\{(x+h)-[x-(n-1)h]\}$$

$$\Delta x^{(n)} = x^{(n-1)}nh.$$

When h = 1,

$$\Delta x^{(n)} = n\,x^{(n-1)}. \quad (1)$$

Note: 1

From equation (1) we get the formula for first order difference, which is obtained by the simple differentiation rule.

For example, $\Delta^2 x^{(n)} = \Delta\left[nhx^{(n-1)}\right] = n\,h\,\Delta\,x^{(n-1)} = n(n-1)h^2\,x^{(n-2)}$ proceeding like this we get

$$\Delta^n x^{(n)} = \Delta\, n(n-1)(n-2)\cdots 1 h^n\, x^0 = n!\,h^n\,.$$

Note: 2

Any polynomial f(x) of degree n can be expressed in the form $f(x) = c_0 x^{(n)} + c_1 x^{(n-1)} + \cdots + c_{n-1} x^{(1)} + c_n$. If f(x) is divided successively by x-0, x-1, x-2, …, x-(n-1), then the remainders give the coefficients $c_n, c_{n-1}, \cdots, c_1, c_0$.

Definition

The reciprocal factorial function $x^{(n)}$ for positive integer n is defined as

$$x^{(-n)} = \frac{1}{(x+h)(x+2h)...(x+nh)}$$

As in the case of factorial function the formula for first order difference of $x^{(-n)}$ is similar to differentiation rule when h = 1.

Theorem:

$\Delta\,x^{(-n)} = (-n)h\,x^{-(n-1)}$. In particular $\Delta\,x^{(-n)} = -n\,x^{-(n+1)}$.

Proof:

$$\Delta\,x^{(-n)} = (x+h)^{(-n)} - x^{(-n)}$$

$$= \frac{1}{(x+2h)(x+3h)\cdots[x+(n+1)h]} + \frac{1}{(x+h)(x+2h)\cdots(x+nh)}$$

$$= \frac{1}{(x+h)(x+2h)\cdots[x+(n+1)h]}\{x+h-[x+(n+1)h]\}$$

$$= \frac{-nh}{(x+h)(x+2h)\cdots[x+(n+1)h]}$$

$$\Delta\,x^{(-n)} = (-n)h\,x^{-(n-1)}.$$

Remark:

$$\Delta^2 x^{(-n)} = \Delta\left[-nhx^{-(n+1)}\right]$$

$$= nh[-(n+1)h]x^{-(n+2)} = (-1)^2\,h^2\,n(n+1)x^{-(n+2)}$$

In general $\Delta^r\,x^{(-n)} = (-1)^r\,h^r\,n(n+1)\cdots(n+r+1)x^{-(n+1)}$.

Example: 1

Form the forward difference table for the following data:

x	0	1	2	3	4
y	8	11	9	15	6

Solution

The difference table is given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	8				
		3			
1	11		-5		
		-2		13	
2	9		8		-36
		6		-23	
3	15		-15		
		9			
4	6				

Example: 2

Find $\Delta (2^x)$.

Solution

$\Delta(2^x) = 2^{x+h} - 2^x$

$\Delta(2^x) = 2^x (2^h - 1)$.

Example: 3

Find the n^{th} difference of e^x.

Solution

$$\Delta(e^x) = e^{x+h} - e^x = e^x(e^h - 1)$$

$$\Delta^2(e^x) = \Delta(\Delta e^x)$$

$$= \Delta[e^x(e^h - 1)]$$

$$= (e^h - 1)\Delta(e^x)$$

$$= (e^h - 1)(e^x)(e^h - 1)$$

$$= e^x(e^h - 1)^2$$

Similarly $\Delta^3(e^x) = e^x(e^h - 1)^3$.

Proceeding like this we get $\Delta^n(e^x) = e^x(e^h - 1)^n$.

Other Difference Operators

The shift operator E and averaging operator μ.

Definition:

The shift operator E is defined by $Ef(x) = f(x+h)$. Hence $E^2 f(x) = Ef(x+h) = f(x+2h)$. In general, for any positive integer n

$$E^n f(x)=f(x+nh)$$

In particular we have

$Ey_0 = y_1$

$E^2 y_0 = y_2$

....

$E^n y_0 = y_n$.

The inverse operator E^{-1} is defined as

$E^{-1}f(x) =f(x-h)$

For any real number n we have

$E^n f(x) =f(x+nh)$.

Note:

$E^m E^n f(x) = E^{m+n} f(x)$

Definition:

The averaging operator is defined as

$$\mu f(x) = \frac{f\left(x+\frac{h}{2}\right)+f\left(x-\frac{h}{2}\right)}{2}$$

There are several relation connecting the operators Δ, ∇, δ, E, μ and the differentiation operator D.

Example 1:

$E\nabla = \nabla E = \Delta$.

Solution:

$E\nabla = E(1 - E^{-1}) E = E - 1 = \Delta$.

Also, $\nabla E = (1 - E^{-1}) E = E - 1 = \Delta$.

Example 2:

$$\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}} = 2+\Delta .$$

Solution:

$$\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}}=\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)E^{\frac{1}{2}}$$

$$= E + 1$$

$$= (1 + \Delta) + 1$$

$$\left(E^{\frac{1}{2}}+E^{-\frac{1}{2}}\right)(1+\Delta)^{\frac{1}{2}}=2+\Delta.$$

1.4 Difference Equations

Any situation in which there exists a sequential relation at discrete values of the independent variable, leads to difference equations. Difference equations may be thought of a discrete counterpart of differential equations and there is a striking similarity between the methods of solving difference equations and differential equations.

Definition

An equation involving the differences of an unknown function y = y(x) at one or more general values of the argument n is called a difference equation.

The following are some examples of difference equations.

$$\Delta y_n + 2y_n = n \quad (1)$$

$$\Delta^2 y_n + 5\Delta y_n + 3y_n = 0 \quad (2)$$

$$\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x \quad (3)$$

We assume that the consecutive values of independent variable differ by unity. With this assumption a difference equation can be written in an alternative form, as illustrated below.

Consider the difference equation, $\Delta y_n + 2y_n = n$.

Since $\Delta y_n = (E-1)y_n = y_{n+1} - y_n$ the above difference equation can be written as $y_{n+1} - y_n + 2y_n = n$ $(i.e)$ $y_{n+1} + y_n = n$.

Consider the difference equation (3), $\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x$.

Since $\Delta u_x = (E-1)u_x = u_{+1} - u_x$ *and* $\Delta^2 u_x = (E-1)^2 u_x = (E^2 - 2E + 1)u_x = u_{x+2} - 2u_{x+1} + u_x$ the difference equation (3) can be written as

$$(u_{x+2} - 2u_{x+1} + u_x) - 4(u_{x+1} - u_x) + 4u_x = 3^x.$$

$$\text{(i.e) } u_{x+2} - 6u_{x+1} + 9u_x = 3^x.$$

Definition

The order of a difference equation is the difference between the largest and smallest subscripts occurring in it, when the equation is expressed in the form free of the difference operator Δ.

The degree of a difference equation, expressed in the form free of Δ, is the higher power of y.

Examples:

1. The difference equation $\Delta y_n + 2y_n = n$, when expressed in a form free of Δ is $y_{n+1} + y_n = n$. The largest subscript in the equation is n+1 and the smallest subscript in the equation is n. Hence the order of the difference equation is 1. The highest power of y is 1. Hence the degree of the difference equation is also 1.
2. The difference equation $\Delta^2 u_x - 4\Delta u_x + 4u_x = 3^x$, when expressed in a form free of Δ is $u_{x+2} - 6u_{x+1} + 9u_x = 3^x$. The largest subscript in the equation is x+2 and the smallest subscript in the equation is x. Hence the order of the difference equation is 2. The highest power of y is 1. Hence the degree of the difference equation is 1.
3. Consider the difference equation $4y_{n+3}^2 - 2y_n y_{n+1} + y_n^2 y_{n+1}^4 = 0$. It is free form Δ. The largest subscript here is n+3 and the smallest subscript is n. Hence the order of the difference equation is 3. The highest power of y is 4 hence the degree of the difference equation is 4.

Note:

The order of the difference equation may not be the highest power of Δ involved in it. For example, consider the equation $\Delta^2 y_n + 2\Delta y_n + y_n = 2^n$. This can be written as

$$(E-1)^2 y_n + 2(E-1)y_n + y_n = 2^n.$$

$$\text{i.e. } E^2 y_n = 2^n$$

$$\text{i.e. } y_{n+2} = 2^n$$

which is not even a difference equation.

Definitions

Solution

A solution of a difference equation is an expression for y_n which satisfies the given difference equation.

General Solution

A solution in which the number of arbitrary constants is equal to the order of the difference equation is called the general solution.

Particular Solution

Any solution which is obtained from the general solution by giving particular values to the arbitrary constants is called a particular solution.

Example: 1

Write the difference equation $\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$ in the subscript notation.

Solution

The given difference equation can be written as

$(E-1)^3 y_x + (E-1)^2 y_x + (E-1)y_x + y_x = 0$

(i.e) $(E^3 - 3E^2 + 3E - 1)y_x + (E^2 - 2E + 1)y_x + (E^2 - 2E + 1)y_x + (E-1)y_x + y_x = 0$

(i.e) $(E^3 - 2E^2 - 4E)y_x = 0$

(i.e) $y_{x+3} - 2y_{x+2} - 4y_{x+1} = 0$.

Example: 2

Find the order of the difference equation $\Delta^3 y_n - 3\Delta^2 y_n + 2\Delta y_n + y_n = \cos \pi x$.

Solution

The given difference equation can be written as

$(E-1)^3 y_n - 3(E-1)^2 y_n + 2(E-1)y_n + y_n = \cos \pi x$

i.e. $(E^3 - 3E^2 + 3E - 1)y_n - 3(E^2 - 2E + 1)y_n + 2(E-1)y_n + y_n = \cos \pi x$

i.e. $(E^3 - 6E^2 + 11E - 5)y_n = \cos \pi x$

$\therefore y_{n+3} - 6y_{n+2} + 11y_{n+1} - 5y_n = \cos\ \pi x$

This difference equation is free from Δ.

$\therefore$ The order of the given difference equation is (n+3) - n = 3.

Example: 3

Show that $y_n = 1 - \frac{2}{n}$ is a solution of the difference equation $(n+1)y_{n+1} + n\,y_n = 2n - 3$.

Solution

$(n+1)y_{n+1} + n\,y_n = (n+1)\left(1 - \frac{2}{n+1}\right) + n\left(1 - \frac{2}{n}\right)$

= n + 1 – 2 + n – 2

$(n+1)y_{n+1} + n\,y_n = 2n-3$.

$\therefore y_n = 1 - \frac{2}{n}$ is a solution of the given difference equation.

Formation of Difference Equations

In the case of differential equations difference equation can be formed by eliminating the constants from the given equation. We can see some examples.

Example: 1

Form the difference equation by eliminating the constant a from $y_n = a3^n$.

Solution

$y_n = a3^n$.

$\therefore y_{n+1} = a3^{n+1} = 3(a3^n) = 3y_n$.

$\therefore y_{n+1} - 3y_n = 0$ is the required difference equation.

Example: 2

Form the difference equation by eliminating the arbitrary constants A and B from the equation $y_n = Aa^n + Bb^n$ where $a \neq b$.

Solution

$$y_n = Aa^n + Bb^n \qquad (1)$$

$\therefore y_{n+1} = Aa^{n+1} + Bb^{n+1}$

$$\text{(i.e) } y_{n+1} = a(Aa^n) + b(Bb^n) \qquad (2)$$

Similarly

$$y_{n+2} = a^2(Aa^n) + b^2(Bb^n) \qquad (3)$$

Eliminating Aa^n and Bb^n from equation (1) and (3) we get

$$\begin{vmatrix} y_n & 1 & 1 \\ y_{n+1} & a & b \\ y_{n+2} & a^2 & b^2 \end{vmatrix} = 0.$$

(i.e) $y_n(ab^2 - a^2b) - y_{n+1}(b^2 - a^2) + y_{n+2}(b-a) = 0$

$\therefore y_{n+2}(b-a) - y_{n+1}(b-a)(b+a) + y_n\,ab(b-a) = 0$

$\therefore y_{n+2} - (a+b)y_{n+1} + ab y_n = 0$ (Since $a \neq b$).

1.5 Linear Difference Equations

The difference equation of the form

$$a_0 y_{n+r} + a_1 y_{n+r-1} + \cdots + a_r y_n = f(n) \qquad (1)$$

where $a_0, a_1, \cdots, a_r$ and f(n) are functions of n is called a linear difference equation.

If $a_0, a_1, \cdots, a_r$ are constants then equation (1) is called a linear difference equation with constant coefficients.

Equation (1) can also be written in the form

$$\left(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r\right) y_n = f(n).$$

In this section**,** deal with linear difference equation with constant coefficients and discuss the methods of solving them. The methods are analogous to the methods of linear differential equations with constant coefficients.

Definition

Consider the difference equation

$$\left(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r\right) y_n = f(n) \qquad (1)$$

Let $\phi_1(n), \phi_2(n), \cdots, \phi_r(n)$ be r independent solutions of

$$\left(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r\right) y_n = 0 \qquad (2)$$

Then $U_n = c_1\phi_1(n) + c_2\phi_2(n) + \cdots + c_r\phi_r(n)$ is the general solution of equation (2) and is called the complementary function (C.F) of equation (1).

If V_n is a particular solution of equation (1) then $y_n = U_n + V_n$ is the complete solution of equation (1). V_n is called the particular integral (P.I) of equation (1).

Thus the complete solution of equation (1) is given by

$$y_n = C.F + P.I.$$

Rules for finding Complementary Function

I. We first consider a linear difference equation of order one given by $y_{n+1} - a y_n = 0$ where a is a constant.

Dividing by a^{n+1} we get, $\frac{y_{n+1}}{a^{n+1}} - \frac{y_n}{a^n} = 0$

i.e. $\Delta\left(\frac{y_n}{a^n}\right) = 0$.

$\therefore \frac{y_n}{a^n} = c$ where c is a constant.

$\therefore y_n - c a^n = 0$ is the solution of the given difference equation.

II. **Consider a linear difference equation of second order given by**

$$\left(E^2 + aE + b\right)y_n = 0 \qquad \textbf{(1)}$$

where a and b are constants.

Then the equation $E^2 + aE + b = 0$ is called the auxiliary equation of equation (1). Let α_1, α_2 be the roots of the auxiliary equation.

Case (i):

α_1 and α_2 are real and distinct.

Equation (1) can be written as $(E - \alpha_1)(E - \alpha_2)\, y_n = 0$.

We can derive two independent solutions of equation (1) by solving the equations $(E - \alpha_1)y_n = 0$ and $(E - \alpha_2)y_n = 0$.

By, I the solution of these equations are $y_n = c_1\alpha_1^n$ and $y_n = c_2\alpha_2^n$ where c_1, c_2 are arbitrary constants. Hence

$$y_n = c_1\alpha_1^n + c_2\alpha_2^n$$

is the general solution of equation (1).

Case (ii):

The roots are real and equal.

Equation (1) takes the form $(E - \alpha_1)^2\, y_n = 0$.

Let $y_n = \alpha_1^n z_n$.

$\therefore (E - \alpha_1)^2 \alpha_1^n z_n = 0$.

(i.e) $\left(E^2 - 2E\alpha_1 + \alpha_1^n\right)\alpha_1^n z_n = 0$

$\therefore \alpha_1^{n+2} z_{n+2} - 2\alpha_1^{n+2} z_{n+1} + \alpha_1^{n+2} z_n = 0$

$z_{n+2} - 2z_{n+1} + z_n = 0$

$\left(E^2 - 2E + 1\right) = 0$

(i.e) $(E - 1)^2 z_n = 0$

$\therefore \Delta^2 z_n = 0$.

$\therefore z_n = c_1 + c_2 n$ where c_1, c_2 are arbitrary constants. Hence

$y_n = (c_1 + c_2 n)\alpha_1^n$ is the general solution.

Case (iii):

The roots are imaginary.

Let the roots be $\alpha + i\beta$ and $\alpha - i\beta$.

$\therefore y_n = c_1(\alpha + i\beta)^n + c_2(\alpha - i\beta)^n$

$= c_1[r(\cos\theta + i\sin\theta)]^n + c_2[r(\cos\theta - i\sin\theta)]^n$ [putting $\alpha = r\cos\theta$ and $\beta = r\sin\theta$]

$= r^n[c_1(\cos n\theta + i\sin n\theta) + c_2(\cos n\theta - i\sin n\theta)]$

$= r^n[A_1\cos n\theta + A_2\sin n\theta]$

where A_1, A_2 are arbitrary constants and $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

$\therefore y_n = r^n[A_1\cos n\theta + A_2\sin n\theta]$ is the general solution where r and θ are given above.

III. Working Rule to find the C. F of the Difference Equation

$$\left(a_0E^r + a_1E^{r-1} + \cdots + a_{r-1}E + a_r\right)y_n = f(n)$$

From the auxiliary equation

$$a_0E^r + a_1E^{r-1} + \cdots + a_{r-1}E + a_r = 0$$

Let $\alpha_1, \alpha_2, ..., \alpha_r$ be its roots.

If $\alpha_1, \alpha_2, ..., \alpha_r$ are all distinct then the C. F is

$$c_1\alpha_1^n + c_2\alpha_2^n + \cdots + c_r\alpha_r^n.$$

If $\alpha_1 = \alpha_2$, the corresponding part of the C.F is $(c_1 + c_2n)\alpha_1^n$.

If $\alpha_1 = \alpha_2 = \alpha_3$, the corresponding part of the C.F is $\left(c_1 + c_2n + c_3n^2\right)\alpha_1^n$.

If there is a part of complex roots $\alpha + i\beta$, $\alpha - i\beta$ the corresponding part of the C.F is $r^n(c_1\cos n\theta + c_2\sin n\theta)$ where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

Rules for finding P.I:

Consider the difference equation $\left(a_0E^r + a_1E^{r-1} + \cdots + a_{r-1}E + a_r\right)y_n = f(n)$.

(i.e) $\phi(E)y_n = f(n)$ where $\phi(E) = a_0E^r + a_1E^{r-1} + \cdots + a_{r-1}E + a_r$.

Type I:

F(n) = a^n where a is a constant.

$\phi(E)a^n = \left(a_0E^r + a_1E^{r-1} + \cdots + a_{r-1}E + a_r\right)a^n$

$= a_0a^{n+r} + a_1a^{n+r-1} + \cdots + a_ra^n$

$= \left(a_0a^r + a_1a^{r-1} + \cdots + a_r\right)a^n$

$= \phi(a)a^n$

$$\therefore \text{P.I} = \frac{1}{\phi(E)} a^n = \frac{a^n}{\phi(a)} \text{ if } \phi(a) \neq 0.$$

Suppose $\phi(a) = 0$.

Then $\phi(a) = (E - a)\, \Psi(E)$.

Let $\frac{1}{E-a} a^n = b_n$

$$\therefore (E-a) b_n = a^n$$

$$b_{n+1} - a\, b_n = a^n$$

$$\therefore \frac{b_{n+1}}{a^{n+1}} - \frac{b_n}{a^n} = \frac{1}{a}$$

$$\therefore \Delta\left(\frac{b_n}{a^n}\right) = \frac{1}{a}$$

$$\therefore \frac{b_n}{a^n} = \frac{n}{a}$$

$$\therefore b_n = n\, a^{n-1}$$

$$\therefore \frac{1}{E-a} a^n = n\, a^{n-1}.$$

By similar argument, we have

$$\frac{1}{(E-a)^2} a^n = \frac{n(n-1)}{2!} a^{n-2}$$

$$\frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)}{3!} a^{n-3}$$

In general,

$$\frac{1}{(E-a)^r} a^n = \frac{n^{(r)} a^{n-r}}{r!}.$$

Type II:

F(n) is a polynomial in n.

$$\text{P.I} = \frac{1}{\phi(E)} [f(n)]$$

$$= \frac{1}{\phi(1+\Delta)} [f(n)]$$

$$\text{P.I} = [\phi(1+\Delta)]^{-1} f(n).$$

We expand $[\phi(1+\Delta)]^{-1}$ in ascending powers of Δ and operate on f(n).

Type III:

f(n) = cos kn or sin kn.

Since coskn = real part of e^{itn} and sinkn = imaginary part of e^{ikn} we can compute $\frac{1}{\phi(E)}e^{ikn}$ by using the formula given in Type I.

Equating real and imaginary part we get the required P.I.

Type IV:

$$f(n) = a^n g(n).$$

$$P.I = \frac{1}{\phi(E)} f(n) = \frac{1}{\phi(E)}\left[a^n g(n)\right]$$

Now,

$$\phi(E)\left[a^n g(n)\right] = \left(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r\right)\left[a^n g(n)\right]$$

$$= a_0 a^{n+r} g(n+r) + a_1 a^{n+r-1} g(n+r-1) + \cdots + a_r a^n g(n)$$

$$= \left[a_0 a^r E^r g(n) + a_1 a^{r-1} E^{r-1} g(n) + \cdots + a_r g(n)\right] a^n$$

$$= a^n \phi(aE) g(n)$$

$$\therefore \frac{1}{\phi(aE)}\left[a^n g(n)\right] = \frac{1}{\phi(E)}\left[a^n g(n)\right] = \frac{1}{\phi(E)} f(n)$$

$$\therefore P.I = a^n \frac{1}{\phi(aE)} g(n).$$

Example: 1

Solve $y_{n+1} - 2y_{n-1} + 2y_{n-1} = 0$.

Solution

The given equation can be written as

$E^2 y_{n+1} - 2Ey_{n-1} + 2y_{n-1} = 0$

$\therefore \left(E^2 - 2E + 2\right) y_{n-1} = 0$

The auxiliary equation is $E^2 - 2E + 2 = 0$.

The roots are $\frac{2 \pm \sqrt{-4}}{2} = 1 \pm i = \alpha + i\beta$ (say)

The complete integral is $y_{n-1} = r^{n-1}\left[A\cos(n-1)\theta + B\sin(n-1)\theta\right]$ where

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{2} \text{ and } \theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

$$\therefore y_{n-1} = \left(\sqrt{2}\right)^{n-1} \left[A\cos\left(\frac{(n-1)\pi}{4}\right) + B\sin\left(\frac{(n-1)\pi}{4}\right) \right].$$

Example: 2

Solve $\left(4E^2 - 4E + 1\right) y_n = 2^n + 2^{-n}$.

Solution

The auxiliary equation is $4E^2 - 4E + 1 = 0$.

(i.e) $(2E - 1)^2 = 0$. Hence the roots are $\frac{1}{2}, \frac{1}{2}$.

The $C.F = (A + Bn)\left(\frac{1}{2}\right)^n$

Particular integral $P.I = \left(\frac{1}{4E^2 - 4E + 1}\right)\left(2^n + 2^{-n}\right)$

Now $\left(\frac{1}{4E^2 - 4E + 1}\right)\left(2^n\right) = \frac{2^n}{4 \times 2^2 - 4 \times 2 + 1} = \frac{2^n}{9}$.

Also $\left(\frac{1}{4E^2 - 4E + 1}\right)\left(2^{-n}\right) = \left(\frac{1}{4E^2 - 4E + 1}\right)\left(\frac{1}{2}\right)^n$

$$= \left(\frac{1}{(2E - 1)^2}\right)\left(\frac{1}{2}\right)^n$$

$$= \frac{n(n-1)}{2!}\left(\frac{1}{2}\right)^{n-2}.$$

Hence the complete solution is $y_n = C.F + P.I.$

$$y_n = (A + Bn)\left(\frac{1}{2}\right)^n + \frac{2^n}{9} + \frac{n(n-1)}{2}\left(\frac{1}{2}\right)^{n-2}.$$

Exercises:

1. Construct the difference table for the following data.

x	45	50	55	60	65	70	75
y	80	95	120	100	85	70	60

2. Form the difference table for the following data:

x	0	1	2	3	4	5
y	1	5	19	55	125	241

and hence find $\Delta^5 y_0$.

3. Form the difference table for the following data:

x	0	1	2	3	4	5	6
y	2	5	8	20	30	10	3

and hence find $\Delta^6 y_0$.

4. Form the difference table of the function y = $x^3 + x^2 - 2x + 1$ for x = -1, 0, 1, 2, 3, 4.
5. Form the difference table for the function y = $x^3 + x + 1$ at x = 0, 1, 2, 3, 4 and hence find (i) $\Delta^2 y_2$ (ii) $\Delta^3 y_1$ (iii) $\Delta^4 y_0$.
6. Find the order and degree of the following difference equations.
   (i) $y_{n-1} - 3y_n = 3^n$ (ii) $y_{n+2} - y_{n+1} + y_n = 0$ (iii) $y_n - y_{n-1} + 6y_{n-2} = 0$.
7. Find the order and degree of the following difference equations.
   (i) $E^2 y_n + 3Ey_{n-1} + y_n = n^2$ (ii) $y_{n+3} + 3y_{n+2} - y_n = n^2 2^n$.
8. Show that $y_n = (A + Bn)2^n$ is a solution of the equation $y_{n+2} - 4y_{n+1} + 4y_n = 0$.
9. Form the difference equation by eliminating a from $y_n = a5^n$.
10. Form the difference equation by eliminating a and b from each of the relations given below.
    (i) $y_n = a2^n + b3^n$ (ii) $y_x = a2^x + b5^x$ (iii) $y_x = a2^x + b(-2)^x$
    (iv) $y_n = (an + b)3^n$.
11. Solve the difference equation $y_{n+2} - 3y_{n+1} + 2y_n = 5^n + 2^n$.
12. Solve the difference equation $u_{n+2} + u_{n+1} + u_n = n^2 + n + 1$.
13. Solve the following difference equations
    (i) $y_{x+2} - 8y_{x+1} + 15y_x = 0$ (ii) $(E^2 - 5E + 6)y_n = 0$ (iii) $(\Delta^2 + 3\Delta + 1)y_n = 0$.

******

UNIT-II

INTERPOLATION

2.1 Introduction

Interpolation

Definition

Interpolation is the process of estimating the value of a function at an intermediate point or the process of finding the value of the function inside the given range is called interpolation.

Interpolation is the process of finding the most appropriate estimate for missing data. It is the "art of reading between the lines of a table". For making the most probable estimate it requires the following assumptions:

(i) The frequency distribution is normal and marked by sudden ups and downs.

(ii) The changes in the series are uniform within a period.

Interpolation technique is used in various disciplines like statistics, economics, business, population studies, price determination etc. It is used to fill in the gaps in the statistical data for the sake of continuity of information. For example, if we know the records for the past five years except the third year which is not available due to unforeseen conditions, the interpolation technique helps to estimate the record for that year too under the assumption that the changes in the records over these five years have been uniform.

Extrapolation

Definition

Extrapolation is the process of finding the values outside the given interval.

It is also possible that we may require information for future in which case the process of estimating the most appropriate value is known as extrapolation.

Given a set of tabular values of a function y=f(x) where the explicit nature of the function is not known, then f(x) is replaced by a simpler function $\phi(x)$ such that f(x) and $\phi(x)$ agree with the set of tabulated points. Any other value may be calculated from $\phi(x)$. This function $\phi(x)$ is known as interpolation function. In particular if $\phi(x)$ is a polynomial then the process is called polynomial interpolation and $\phi(x)$ is called an interpolating polynomial. The existence of an interpolating polynomial is supported by Weierstras's approximation theorem which asserts that any continuous function on a closed interval can be approximated by a polynomial.

2.2 Lagrange's interpolation formula for unequal intervals

Let y=f(x) be the function such that f(x) is taking the values $y_0, y_1, \ldots, y_n$ corresponding to $x = x_0, x_1, \ldots, x_n$.

In the case of the values of independent variable are not equally spaced and when the differences of dependent variable are not small, we will use Lagrange's interpolation formula.

Let f(x) be a polynomial in x of degree n. Lagrange's interpolation formula for unequal intervals is

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_1)\cdots(x_0-x_n)} f(x_0) +$$
$$\frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f(x_1) + \cdots +$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)(x_n-x_2)\cdots(x_n-x_{n-1})} f(x_n).$$

Example: 1

Using Lagrange's interpolation formula, find the value corresponding to x=10 from the following table:

x	5	6	9	11
y	12	13	14	16

Solution

Given $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $x = 10$

$Y_0 = f(x_0) = 12$

$Y_1 = f(x_1) = 13$

$Y_2 = f(x_2) = 14$

$Y_3 = f(x_3) = 16$

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$= \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) +$$

$$\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$= \frac{4.1.1}{1.4.6}(12) + \frac{5.1.(-1)}{1.3.5}(13) + \frac{5.4.1}{4.3.2}(14) + \frac{5.4.1}{4.3.2}(14) + \frac{5.4.1}{6.5.2}(16)$$

= 14.63.

Y=f(x) = f(10) = 14.63.

Example: 2

Find the polynomial f(x) by using Lagrange's formula and hence find f(3) for

x	0	1	2	5
f(x)	2	3	12	147

Solution

Given $x_0 = 0$; $x_1 = 1$; $x_2 = 2$; $x_3 = 5$

$y_0 = f(x_0) = 2$

$y_1 = f(x_1) = 3$

$y_2 = f(x_2) = 12$

$y_3 = f(x_3) = 147$

The Lagrange's formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_1)(x_0-x_3)} f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f(x_2) + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f(x_3)$$

$$y = f(x) = \frac{(x-1)(x-1)(x-5)}{(0-1)(0-2)(0-3)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3) +$$

$$\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$= \frac{(x-1)(x-1)(x-5)}{-6}(2) + \frac{x(x-2)(x-5)}{4}(3) +$$

$$\frac{x(x-1)(x-5)}{-6}(12) + \frac{x(x-1)(x-2)}{60}(147)$$

Which is the polynomial of y=f(x)

$$y = f(3) = \frac{(3-1)(3-1)(3-5)}{-6}(2) + \frac{3(3-2)(3-5)}{4}(3) +$$

$$\frac{3(3-1)(3-5)}{-6}(12) + \frac{3(3-1)(3-2)}{60}(147)$$

y = f(3) = 44.5.

2.3 Inverse interpolation by Lagrange's method

The process of finding a value of x for the corresponding value of y is called inverse interpolation. In such a case, we will take y as independent variable and x as dependent variable.

Therefore the Lagrange's inverse interpolation formula is

$$x = f(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}.f(y_0) +$$

$$\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}.f(y_1) + \cdots +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})}.f(y_n).$$

Example: 1

Find the value of x, corresponding to y = 100 from the following table:

x	3	5	7	9	11
y	6	24	58	108	174

Solution

Given $y_0 = 6$, $y_1 = 24$, $y_2 = 58$, $y_3 = 108$, $y_4 = 174$ also, y = 100

x = f(y);

$x_0 = f(y_0) = 3$, $x_1 = f(y_1) = 5$, $x_2 = f(y_2) = 7$, $x_3 = f(y_3) = 9$

The Lagrange's formula for inverse interpolation is

$$x = f(y) = \frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}.f(y_0) +$$

$$\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}.f(y_1) + \cdots +$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})}.f(y_n)$$

= 0.3534 – 1.5155 + 2.8870 + 7.0676 – 0.1369

x = f(y) = 8.6556.

Example: 2

Find the value of x when y = 85 using Lagrange's formula for the table

x	2	5	8	14
y	94.8	87.9	81.3	68.7

Solution

Given y_0 = 94.8, y_1 = 87.9, y_2 = 81.3, y_3 = 68.7, also, y = 85

x = f(y);

$x_0 = f(y_0) = 2$, $x_1 = f(y_1) = 5$, $x_2 = f(y_2) = 8$, $x_3 = f(y_3) = 14$

The Lagrange's formula for inverse interpolation is

$$x=f(y)=\frac{(y-y_1)(y-y_2)\cdots(y-y_n)}{(y_0-y_1)(y_0-y_2)\cdots(y_0-y_n)}.f(y_0)+$$

$$\frac{(y-y_0)(y-y_2)\cdots(y-y_n)}{(y_1-y_0)(y_1-y_2)\cdots(y_1-y_n)}.f(y_1)+\cdots+$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)\cdots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)(y_n-y_2)\cdots(y_n-y_{n-1})}.f(y_n)$$

= -0.1438778 + 3.3798011 + 3.3010599 – 0.2331532

= 6. 3038.

Therefore the value of x when y=85 is 6.3038.

2.4 Newton's Divided Difference

Let the function y = f(x) takes the values $f(x_0)$, $f(x_1)$, …, $f(x_n)$ corresponding to the values x_1-x_0, $x_2 - x_1$, $x_3 - x_2$, …, x_n-x_{n-1} need not to be equal.

Be first divided difference of f(x) for the argument x_0, x_1 is defined as $\frac{f(x_1)-f(x_0)}{x_1-x_0}$

it is denoted by $f(x_0, x_1)$ or $[x_0, x_1]$ or $\Delta f(x_0)$.

(i.e) $f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}$.

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)
x_0	$f(x_0)$			
x_1	$f(x_1)$	$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$		
			$f(x_0, x_1, x_2) = \frac{f(x_2 x_1) - f(x_1 x_0)}{x_2 - x_0}$	$f(x_0, x_1, x_2, x_3) = \frac{f(x_1 x_2 x_3) - f(x_0 x_1 x_2)}{x_3 - x_0}$
x_2	$f(x_2)$	$f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$	$f(x_1, x_2, x_3) = \frac{f(x_3 x_2) - f(x_2 x_1)}{x_3 - x_1}$	$f(x_1, x_2, x_3, x_4) = \frac{f(x_2 x_3 x_4) - f(x_1 x_2 x_0)}{x_4 - x_1}$
		$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_3 - x_2}$	$f(x_2, x_3, x_4) = \frac{f(x_4 x_3) - f(x_3 x_2)}{x_4 - x_1}$	
x_3	$f(x_3)$			
		$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_4 - x_3}$		
x_4	$f(x_4)$			

Fourth divided difference is $\Delta^4 f(x) = f(x_0, x_1, x_2, x_3, x_4) = \frac{f(x_1 x_2 x_3 x_4) - f(x_0 x_1 x_2 x_3)}{x_4 - x_0}$.

Properties of Divided Differences

Property: 1

The value of any divided difference is independent of the order of the arguments. That is, the divided differences are symmetrical in all their arguments.

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = f(x_1, x_0) \qquad (1)$$

Again, $$f(x_0,x_1)=\frac{f(x_0)}{x_0-x_1}-\frac{f(x_1)}{x_0-x_1}=\frac{f(x_0)}{x_0-x_1}+\frac{f(x_1)}{x_1-x_0} \tag{2}$$

In the same way, $$f(x_1,x_0)=\frac{f(x_1)}{x_1-x_0}+\frac{f(x_0)}{x_0-x_1} \tag{3}$$

From equation (2) and equation (3), we have $f(x_0,x_1)=f(x_1,x_0)$.

Similarly,

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$=\frac{1}{x_2-x_0}\left[\left(\frac{f(x_1)}{x_1-x_2}+\frac{f(x_2)}{x_2-x_1}\right)-\left(\frac{f(x_0)}{x_0-x_1}+\frac{f(x_1)}{x_1-x_0}\right)\right]$$

$$=\frac{1}{x_2-x_0}\left[\left(\frac{1}{x_1-x_2}-\frac{1}{x_1-x_0}\right)f(x_1)+\frac{f(x_2)}{x_2-x_1}-\frac{f(x_0)}{x_0-x_1}\right]$$

$$=\frac{1}{x_2-x_0}\left[\frac{x_2-x_0}{(x_1-x_2)(x_1-x_0)}f(x_1)+\frac{f(x_2)}{x_2-x_1}-\frac{f(x_0)}{x_0-x_1}\right]$$

$$f(x_0,x_1,x_2)=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)} \tag{4}$$

From equation (4), we find

$$f(x_0,x_1,x_2)=f(x_1,x_0,x_2)=f(x_1,x_2,x_0)=\cdots$$

This shows that $f(x_0,x_1,x_2)$ is independent of the order of the arguments.

By mathematical induction, we can prove that

$$f(x_0,x_1,x_2,\cdots,x_n)=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}+\cdots$$
$$+\frac{f(x_n)}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}.$$

This is symmetrical we know that any two arguments. Therefore, the divided differences are symmetrical we know that any two arguments.

Property: 2

The operator Δ is linear.

Proof:

If f(x) and g(x) are two functions α and β are constants, then

$$\Delta[\alpha f(x)+\beta g(x)]=\frac{[\alpha f(x_1)+\beta g(x_1)][\alpha f(x_0)+\beta g(x_0)]}{x_1-x_0}$$

$$= \alpha \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \beta \frac{g(x_1) - g(x_0)}{x_1 - x_0}$$

$$\Delta[\alpha f(x) + \beta g(x)] = \alpha \, \Delta \, f(x) + \beta \Delta g(x).$$

Corollary: 1

Setting $\alpha = \beta = 1$, $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$.

Corollary: 2

Setting $\beta = 0$, $\Delta[\alpha f(x)] = \alpha \Delta f(x)$.

Property: 3

The n^{th} divided differences of a polynomial of degree n are constants.

Proof:

Taking $f(x) = x^n$ where n is a positive integer,

$$f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{x_1^n - x_0^n}{x_1 - x_0}$$

$$= x_1^{n-1} + x_0 \, x_1^{n-2} + x_0^2 \, x_1^{n-3} + \cdots + x_0^{n-1}$$

= a polynomial function of degree (n-1) and symmetrical in x_0, x_1 with leading coefficient 1.

Again,

$$f(x_0, x_1 x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

$$= \frac{\left(x_2^{n-1} + x_1 \, x_2^{n-2} + \cdots + x_1^{n-1}\right) - \left(x_0^{n-1} + x_1 \, x_0^{n-2} + \cdots + x_1^{n-1}\right)}{x_2 - x_0}$$

$$= \frac{x_2^{n-1} - x_0^{n-1}}{x_2 - x_0} + \frac{x_1\left(x_2^{n-2} - x_0^{n-2}\right)}{x_2 - x_0} + \cdots + \frac{x_1^{n-2}(x_2 - x_0)}{x_2 - x_0}$$

$$= \left(x_2^{n-2} + x_0 \, x_2^{n-3} + \cdots + x_0^{n-2}\right) + x_1\left[x_2^{n-3} + x_0 \, x_2^{n-4} + x_0^{n-3}\right] + \cdots + x_1^{n-2}$$

= a polynomial of degree (n - 2) and symmetrical i x_0, x_1, x_2 with leading coefficient 1.

Proceeding in this way, the r^{th} divided differences of x^n will be a polynomial of degree (n-r) and symmetrical in $x_0, x_1, x_2, ..., x_r$ with leading coefficient 1.

Hence n^{th} order divided differences of x^n will be a polynomial of degree n – n =0 , with leading coefficient 1. That is, its value is 1.

That is $\Delta^n x^n = 1$.

$\Delta^{n+i} x^n = 0$, for $i = 1,2,...$

Hence, $\Delta^n \left[a_0 x^n + a_1 x^{n-1} + \cdots a_n\right]$

$$= a_0 \Delta^n x^x + a_1 \Delta^n x^{n-1} + \cdots + \Delta^n a_n$$

$$= a_0 \, 1 + 0 + 0 + \cdots + 0 = a_0 .$$

Note:

Conversely, if the n^{th} divided difference of a polynomial is constant, then the polynomial is degree of n.

Relation between Divided Differences and Forward Differences

If the arguments x_0, x_1, x_2, ... are equally spaced, then we have, $x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = \cdots x_n - x_{n-1} = h$.

$$\Delta f(x_0) = f(x_1, x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta f(x_0)}{h}$$

$$\Delta^2 f(x_0) = \frac{\Delta f(x_1) - \Delta f(x_0)}{x_2 - x_0} = \frac{\frac{1}{h}\Delta f(x_1) - \frac{1}{h}\Delta f(x_0)}{2^h}$$

$$= \frac{1}{2h^2} \Delta^2 f(x_0)$$

Similarly,

$$\Delta^3 f(x_0) = \frac{\Delta^3 f(x_0)}{3! h^3}$$

$$\Delta^n f(x_0) = \frac{\Delta^n f(x_0)}{n! h^n} .$$

2.5 Newton's divided difference interpolation formula for unequal intervals

$$y = f(x) = f(x_0) + (x = x_0) f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3) + \cdots$$

$$+ (x - x_0)(x - x_1)(x - x_2) \cdots (x - x_{n-1}) f(x_0, x_1, x_2 x_3, \cdots x_n) .$$

Example: 1

Using Newton's divided difference formula, find the values of f(2), f(8) and f(15) given the following table:

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

Solution

We form the divided difference table since the intervals are unequal.

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)	Δ^4f(x)
4	48				
5	100	$\frac{100-48}{5-4}=52$			
			$\frac{97-52}{7-4}=15$		
7	294	$\frac{294-100}{7-5}=97$		$\frac{21-15}{10-4}=1$	
			$\frac{202-97}{10-5}=21$		0
10	900	$\frac{900-294}{10-7}=202$		$\frac{27-21}{11-5}=1$	
			$\frac{310-202}{11-7}=27$		0
11	1210	$\frac{1210-900}{11-10}=310$		$\frac{33-27}{13-7}=1$	
			$\frac{409-310}{13-10}=33$		
		$\frac{2028-1210}{13-11}=409$			
13	2028				

By Newton's Divided Difference interpolation formula is

$$y = f(x) = f(x_0) + (x = x_0)f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3)$$

Here $x_0 = 4$, $x_1 = 5$, $x_2 = 7$, $x_3 = 10$, $x_4 = 11$, $x_5 = 13$

Also, $f(x_0) = 48$, $f(x_0, x_1) = 52$, $f(x_0, x_1, x_2) = 15$, $f(x_0, x_1, x_2 x_3) = 1$

y = f(x) = 48 +(x-4)(52)+(x-4)(x-5)(15)+(x-4)(x-5)(x-7).1

f(2) = 48+(2-4)(52)+(2-4)(2-5)(15)+(2-4)(2-5)(2-7)

f(2) = 4

f(8) = 48+(4)(52)+(4)(3)(15)+(4)(3)(1)

f(8) = 448

f(15) = 48 + 11 (52) + (11)(10)(15) + (11)(10)(8)

f(15) = 3150.

Example: 2

Using Newton's divided difference formula, find u(3) given that u(1) = -26, u(2) = 12, u(4) = 256, u(6) = 844.

Solution

We form the divided difference table since the intervals are unequal.

x	f(x)	Δf(x)	Δ^2f(x)	Δ^3f(x)

1	-26			
		$\frac{12+26}{2-1}=38$		
2	12		$\frac{122-38}{4-1}=28$	
		$\frac{256-12}{4-2}=122$		$\frac{43-28}{6-1}=3$
4	256		$\frac{294-122}{6-2}=43$	
		$\frac{844-256}{6-4}=294$		
6	844			

Newton divided difference interpolation formula is

$$y = f(x) = f(x_0) + (x = x_0) f(x_0, x_1) + (x - x_0)(x - x_1).f(x_0, x_1, x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2).f(x_0, x_1, x_2 x_3)$$

Here,

$$y = u(x) = u(x_0) + (x - x_0)\, u(x_0,x_1) + (x - x_0)(x - x_1)\, u(x_0,x_1,x_2)$$

$$+ (x - x_0)(x - x_1)(x - x_2)\, u(x_0,x_1,x_2,x_3)$$

Here,

$x_0 = 1$, $x_1 = 2$, $x_2 = 4$, $x_3 = 6$

$u(x_0) = -26$, $u(x_0,x_1) = 38$, $u(x_0,x_1,x_2) = 28$, $u(x_0,x_1,x_2,x_3) = 3$

$u(x) = -26 + (x-1)(38) + (x-1)(x-2)(28) + (x-1)(x-2)(x-3)(3)$

for $x = 3$,

$y = u(x) = -26 + (3-1)(38) + (3-1)(3-2)(28) + (3-1)(3-2)(3-4)(3)$

$= -26 + 76 + 56 - 6$

$u(3) = 100$.

Newton's Forward and Backward Interpolation Formula for Equal Intervals

Newton's Forward Interpolation Formula

$$y = f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots$$

where $p = \frac{x - x_0}{h}$, h is the width of interval

$x = x_0 + ph$.

Newton's Forward Interpolation Formula

$$y = f(x) = y_0 + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

where $p = \frac{x - x_n}{h}$, h is the width of interval

$x = x_n + ph$.

Example: 1

Using Newton's forward interpolation formula, find the polynomial f(x) satisfying the following data. Hence evaluate y at x = 5.

x	4	6	8	10
y	1	3	8	10

Solution

We form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
4	1			
		3-1=2		
6	3		5-2=3	
		8-3=5		-3-3=-6
8	8		2-5=-3	
		10-8=2		
10	10			

There are only 4 data given. Hence the polynomial will be degree 3.

Therefore Newton's –Gregory Forward interpolation Formula is

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Here $y_0 = 1$; $p = \frac{x - x_0}{h} = \frac{x-4}{2}$

$$y = f(x) = 1 + \frac{\left(\frac{x-4}{2}\right)}{1!}(2) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)}{2!}(3) + \frac{\left(\frac{x-4}{2}\right)\left(\frac{x-4}{2}-1\right)\left(\frac{x-4}{2}-2\right)}{3!}(-6)$$

$$\therefore y = f(x) = \frac{1}{8}\left[-x^3 + 21x^2 - 126x + 240\right]$$

When x= 5,

$$\therefore y = f(5) = \frac{1}{8}\left[-(5)^3 + 21(5)^2 - 126(5) + 240\right] = 1.25$$

Y = 1.25 when x = 5.

Example: 2

A third degree polynomial passes through the points (0,-1), (1,1), (2,1) and (3,-2) using Newton's forward interpolation formula find the polynomial. Hence find the value at 1.5.

Solution

We form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	-1			
		1+1 = 2		
1	1		0-2=-2	
		1-1=0		-3+2=-1
2	1		-3-0=-3	
		-2-1=-3		
3	-2			

There are only 4 data given. Hence the polynomial will be degree 3.

Therefore Newton's –Gregory Forward interpolation Formula is

$$y = f(x) = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$

Here y_0 = -1; $p = \frac{x - x_0}{h} = \frac{x-0}{1} \Rightarrow p = x$

$$y = f(x) = -1 + \frac{x}{1!}\times 2 + \frac{x(x-1)}{2!}(-2) + \frac{x(x-1)(x-2)}{3!}(-1)$$

$$\therefore y = f(x) = -\frac{1}{6}\left[x^3 + 3x^2 - 16x + 6\right]$$

When x= 1.5,

$$\therefore y = f(1.5) = -\frac{1}{6}\left[(1.5)^3 + 3(1.5)^2 - 16(1.5) + 6\right] = 1.3125$$

Y = 1.3125 when x = 1.5.

Example: 3

Use Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data: f(-0.75) = - 0.07181250; f(-0.5) = -0.024750; f(-0.25) = 0.33493750; f(0) = 1.10100. Hence find f(-1/3).

Solution

We form the difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
-0.75	-0.07181250			
		0.0470625		
			0.312625	
-0.50	-0.024750			0.09375
		0.3596875		
			0.400375	
-0.25	0.33493750			
		0.7660625		
0	1.10100			

Newton's backward difference formula is

$$y = f(x) = y_3 + \frac{p}{1!}\nabla y_3 + \frac{p(p+1)}{2!}\nabla^2 y_3 + \frac{p(p+1)(p+3)}{3!}\nabla^3 y_3$$

where $p = \frac{x - x_3}{h}$; p = 4x.

$$\therefore y = f(x) = 1.10100 + \frac{4x}{1!}(0.7660625) + \frac{4x(4x+1)}{2!}(0.406375) + \frac{4x(4x+1)(4x+2)}{3!}$$

$$y = f(x) = x^3 + 4.001x^2 + 4.002x + 1.101$$

$$\therefore f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 + 4.001\left(-\frac{1}{3}\right)^2 + 4.002\left(-\frac{1}{3}\right) + 1.101 = 0.174518518.$$

Exercises:

1. Using Lagrange's interpolation formula, find f(4) given that f(0) = 2, f(1) = 3, f(2) 12, f(15) = 3587.
2. Find the third degree polynomial f(x) satisfying the following data. Also, find f(4), f(6).

x	1	3	5	7
y	24	120	336	720

3. Using Lagrange's interpolation find y(2) from the following data

x	0	1	3	4	5
y	0	1	81	256	625

4. Apply Lagrange's inverse formula to obtain the root of equation f(x) = 0. Given that, f(0) = -4; f(1) = 1; f(3) = 29; f(4) 52.

5. Find f(x) as a polynomial in x for the following data by Newton's divided difference formula:

x	-4	-1	0	2	5
f(x)	1245	33	5	9	1335

6. Using Newton's Divided difference formula, fit a polynomial to the data and hence find y chosen x = 1.

x	-1	0	2	3
y	-8	3	1	12

7. If $f(x) = \frac{1}{x}$, find f(a, b, c, d) or $\Delta^3\left(\frac{1}{a}\right)$ (or) $\Delta^3\left(\frac{1}{a}\right) = -\frac{1}{abcd}$

8. Using Newton's forward interpolation formula, find the polynomial satisfying the following data. Hence find f(x).

x	0	5	10	15
y	14	379	1444	3584

9. Use Newton's forward interpolation formula find the cubic polynomial which takes places the following values:

x	0	1	2	3
y	1	2	1	10

10. State Lagrange's interpolation formula.
11. What is the Lagrange's formula to find y if there sets of values (x_0, y_0), (x_1, x_2) and (y_1, y_2) are given.
12. What is the assumption we make when Lagrange's formula is used?
13. Give the inverse of Lagrange's interpolation formula.
14. Using the Newton's divided difference formula, find the missing value from the table:

x	1	2	4	5	6
y	14	15	5	-	9

*****

UNIT – III

NUMERICAL DIFFERENTIATION AND INTEGRATION

3.7 Waddle's Rule

3.8 Cote's Method

3.1 Introduction

We assume that a function f(x) is given in a tabular form at a set of n+1 distinct points x_0, x_1, …, x_n. From the given tabular data, we require approximations to the derivatives $f^{(r)}(x'), r \geq 1$, where x' may be a tabular or a non-tabular point. We consider the cases r = 1, 2.

In many applications of science and engineering, we require to compute the value of the definite integral $\int_a^b f(x)\,dx$, where f(x) may be given explicitly or as a tabulated data. Even when f(x) is given explicitly, it may be a complicated function such that integration is not easily carried out.

Here, we shall derive numerical methods to compute the derivatives or evaluate an integral numerically.

3.2 Numerical Differentiation

Approximation to the derivatives can be obtained numerically using the following two approaches

(i) Methods based on finite differences for equispaced data.

(ii) Methods based on divided differences or Lagrange interpolation for non-uniform data.

Numerical differentiation is the process of calculating the derivation of a given function by means of a table of given values of that function. That is, if (x_i, y_i) are the given set of values, then the process of computing the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \text{etc.}$ at a given point is called numerical representation.

The interpolation formula depends on the particular value of x at which the derivatives are required.

(i) If the values of x are not equally spaced, we represent the function by Newton's divided difference formula and the derivatives are obtained.

(ii) If the values of x are equally spaced the derivatives are calculated by using Newton's Forward or backward interpolation formula.

3.3 Numerical Integration

The process of evaluating a definite integral $\int_a^b f(x)\,dx$ from a set of tabulated values (x_i, y_i); i=0, 1, …, n of the integrand y = f(x) is called numerical integration.

Newton cote's formula (or) General Quadrature formula for equidistant coordinates

Let $I = \int_a^b y\,dx$ where y = f(x) takes the values $y_0, y_1, \ldots, y_n$ for $x_0, x_1, \ldots, x_n$. Let us divide the interval (a, b) into n sub intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$, …, $x_n = x_0+nh = b$. After simplification, we get

$$I = \int_{x_0}^{x_n} y dx = nh\left[y_0 + \frac{n}{2}\Delta y_0 + \frac{n(2n-3)}{12}\Delta^2 y_0 + \frac{n(n-2)^2}{24}\Delta^3 y_0 + \cdots \right] \qquad (1)$$

which is the general quadrature formula.

By putting n = 1 in equation (1), Trapezoidal rule is obtained.

By putting n = 2 in equation (1), Simpson's $\frac{1}{3}$ rule is obtained.

By putting n = 3 in equation (1), Simpson's $\frac{3}{8}$ rule is obtained.

3.4 Trapezoidal Rule

Putting n = 1 in equation (1), we get

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_0+h} y\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0 \right]$$

$$= \frac{h}{2}[2y_0 + \Delta y_0]$$

$$= \frac{h}{2}[y_0 + (y_0 + \Delta y_0)]$$

$$= \frac{h}{2}[y_0 + y_1]$$

Similarly,

$$\int_{x_1}^{x_2} y\,dx = \int_{x_0+h}^{x_0+2h} y\,dx = h\left[y_1 + \frac{1}{2}\Delta y_1 \right]$$

$$= \frac{h}{2}[2y_1 + \Delta y_1]$$

$$= \frac{h}{2}[y_1 + (y_1 + \Delta y_1)]$$

$$= \frac{h}{2}[y_1 + y_2]$$

and so on.

$$\int_{x_{n-1}}^{x_n} y\,dx = \int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}[y_{n-1} + y_n]$$

$\therefore$ Adding all the above, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$ which is called Trapezoidal Rule.

3.5 Simpson's $\frac{1}{3}$ Rule (or) Simpson's Rule

Putting n = 2 in equation (1) and neglecting the differences of higher order than second order. We get,

$$\int_{x_0}^{x_n} y\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})].$$

Note:

It should be noted that for applying this rule, the interval must be divided into even number of sub intervals of width h.

3.6 Simpson's $\frac{3}{8}th$ Rule

Putting n = 3 in equation (1) and neglecting the higher order differences above the third, we get

$$\int_{x_0}^{x_n} y\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

Note:

This is not as accurate as Simpson's rule. This rule is used when the number of subintervals is a multiple of 3.

Example: 1

Using Trapezoidal rule, evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ taking 8 intervals.

Solution

Given $y = f(x) = \frac{1}{1+x^2}$

Given x = -1 to 1.

$\therefore$ Length of interval = 2.

$\therefore h = \frac{2}{8} = 0.25$.

$\because h = \frac{b-a}{number\ of\ intervals}$

$= \frac{1-(-1)}{8} = 0.25$

$\therefore$ We form a table

x	-1	-0.75	-0.5	-0.25	0	0.25	0.50	0.75	1
y	0.5	0.64	0.8	0.9412	1	0.9412	0.8	0.64	0.5

$\therefore$ Trapezoidal rule is $\int_{x_0}^{x_n} y\ dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$

$$\int_{-1}^{1} \frac{1}{1+x^2}\ dx = \frac{0.25}{2}[(0.5+0.5) + 2(0.64+0.8+0.9412+1+0.9412+0.8+0.64)]$$

$$= \frac{0.25}{2}[1 + 2(5.7624)]$$

=1.5656.

Example: 2

Evaluate $\int_0^1 \frac{dx}{1+x^2}$ with $h = \frac{1}{6}$ by Trapezoidal rule.

Solution

Given $y = f(x) = \frac{1}{1+x^2}$

Given x = 0 to 1

Also $h = \frac{1}{6}$.

The table is

x	0	1/6	2/6	3/6	4/6	5/6	1
y	1	36/37	9/10	4/5	9/13	36/61	1/2

By Trapezoidal rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

$$\int_0^1 \frac{1}{1+x^2}\ dx = \frac{\left(1/6\right)}{2}\left[\left(1+\frac{1}{2}\right) + 2\left(\frac{36}{37} + \frac{9}{10} + \frac{4}{5} + \frac{9}{13} + \frac{36}{31}\right)\right]$$

$$= \frac{1}{12}\left[\frac{3}{2} + 2(3.9554)\right]$$

= 0.7842.

Example: 3

Evaluate $\int_4^{5.2} \log e^x\ dx$ by using (i) Trapezoidal rule (ii) Simpson's rule (iii) Simpson's $\frac{3}{8}$ rule, given that h = 0.2.

Solution

Given y = f(x) = log e^x

x = 4 to 5.2, h = 0.2.

The table is

x	4	4.2	4.4	4.6	4.8	5.0	5.2
y	1.737	1.824	1.910	1.997	2.084	2.171	2.258

(i) By Trapezoidal rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \cdots + y_{n-1})]$$

$$= \frac{0.2}{2}[(1.737 + 2.258) + 2(1.824 + 1.910 + 1.997 + 2.084 + 2.171)]$$

= 0.3967.

(ii) By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

$$\int_4^{5.2} \log e^x\ dx = \frac{0.2}{3}[(1.737 + 2.258) + 4(1.824 + 1.997 + 2.171) + 2(1.910 + 2.084)]$$

= 0.0666[3.995+4(5.992)+2(3.994)]

= 2.394.

(iii) By Simpson's $\frac{3}{8}$ rule,

$$\int_{x_0}^{x_n} y\ dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 \cdots + y_{n-1}) + 2(y_3 + y_6 + \cdots + y_{n-3})]$$

$$= \frac{3(0.2)}{8}[(1.737 + 2.258) + 3(1.824 + 1.910 + 2.084 + 2.171) + 2(1.997)]$$

= 2.3967.

Romberg's Method

Romberg's method is used to evaluate the integral of the form $I = \int_a^b y\,dx$.

For Romberg's method, let us apply Trapezoidal rule several times find the value of I's as follows:

I_1: Dividing 'h' into 2 parts (i.e) $\frac{h}{2}$

I_2: Dividing 'h' into 4 parts (i.e) $\frac{h}{4}\left(\frac{h/2}{2}\right)$

I_3: Dividing 'h' into 8 parts (i.e) $\frac{h}{8}\left(\frac{h/4}{2}\right)$

I_4: Dividing 'h' into 16 parts (i.e) $\frac{h}{16}\left(\frac{h/8}{2}\right)$

and so on.

Applying Romberg's formula,

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

For I_1, I_2; I_2, I_3; I_3, I_4; …

We get the values of I. This method continues till we get two successive values of I's are equal. The systematic refinement of the values of I's is called Romberg's method.

Example: 1

Use Romberg's method to compare $\int_0^1 \frac{dx}{1+x^2}$. Correct to 4 decimal places and hence find an approximate value of π.

Solution

Let $I = \int_0^1 \frac{dx}{1+x^2}$

To find I_1:

Dividing h into 2 parts (i.e) $\frac{h}{2}$.

$$\therefore h = \frac{1-0}{2} = 0.5$$

X	0	0.5	1
y = $\frac{1}{1+x^2}$	1	0.8	0.5

By Trapezoidal rule,

$$I_1 = \frac{h}{2}[(y_0 + y_2) + 2(y_1)]$$

$$= \frac{0.5}{2}[(1+0.5) + 2(0.8)]$$

$$= 0.775.$$

To find I_2:

Dividing h into 4 parts (i.e) $\frac{h}{4}$.

$\therefore h = \frac{1-0}{4} = 0.25$

x	0	0.02	0.50	0.75	1
y	1	0.941	0.8	0.64	0.5

By Trapezoidal rule,

$$I_2 = \frac{0.25}{2}\left[(1+0.5)+2(0.941+0.8+0.64)\right]$$

$= 0.7828.$

To find I_3:

Dividing h into 8 parts (i.e) $\frac{h/4}{2}$

$\therefore h = \frac{1-0}{8} = 0.125$

x	0	0.125	0.25	0.375	0.5	0.625	0.75	0.875	1
y	1	0.9846	0.9412	0.8767	0.80	0.7191	0.64	0.5664	0.5

By Trapezoidal rule,

$$I_3 = \frac{0.125}{2}\left[(1+0.5)+2(0.984+0.94+0.876+0.8+0.719+0.64+0.56)\right]$$

$= 0.78475.$

By Romberg's Formula:

Iteration 1:

$$I = I_2 + \left(\frac{I_2 - I_1}{3}\right)$$

$$= 0.7828 + \left(\frac{0.7828 - 0.775}{3}\right)$$

I = 0.7854.

Iteration 2:

$$I = I_3 - \left(\frac{I_3 - I_2}{3} \right)$$

$$= 0.78475 - \left(\frac{0.78475 - 0.7828}{3} \right)$$

I = 0.7854.

From the last two iterations, I = 0.7854.

To find π:

$$\int_0^1 \frac{dx}{1+x^2} = \left[\tan^{-1}\left(\frac{x}{1} \right) \right]_0^1$$

$$= \tan^{-1}(1) - \tan^{-1}(0)$$

$$= \frac{\pi}{4}.$$

But $\int_0^1 \frac{dx}{1+x^2} = 0.7854$

$$0.7854 = \frac{\pi}{4}$$

π = 4(0.7854)

$$\pi \approx 3.1416.$$

Two points and three points Gaussian Quadrature formulas

Gaussian Quadrature

Gauss derived a formula which is used to evaluate the integral of the form $\int_{-1}^{1} F(u)\, du$.

One point Gaussian Formula

The one point Gaussian formula is given by $\int_{-1}^{1} f(x)dx = 2.f(0)$which is exact for polynomials of degree up to 1.

Two point Gaussian Formula

The two point Gaussian formula is $\int_{-1}^{1} f(x)dx = f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$ and this is exact for polynomial of degree up to 3.

Three point Gaussian Formula

The three point Gaussian formula is $\int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$ which is exact for polynomial of degree up to 5.

Note:

(i) The integral $\int_{a}^{b} F(t)\, dt$, can be transformed into $\int_{-1}^{1} f(x)dx$ by the linear transformation.

$$t = \left[\frac{(b-a)x + (b+a)}{2}\right].$$

(ii) The integral $\int_{a}^{b} F(x)\, dx$, can be transformed into $\int_{-1}^{1} f(t)dt$ by the linear transformation.

$$x = \left[\frac{(b-a)t + (b+a)}{2}\right].$$

Example: 1

Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ by two point and three point Gaussian formula and compare with exact value.

Solution

By two point Gaussian formula, $\int_{-1}^{1} f(x)dx = f\left(-\sqrt{\frac{1}{3}}\right) + f\left(\sqrt{\frac{1}{3}}\right)$

Given $f(x) = \frac{1}{1+x^2}$.

$$\therefore f\left(-\sqrt{\frac{1}{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$$

$$\therefore f\left(\sqrt{\frac{1}{3}}\right) = \frac{1}{1+\frac{1}{3}} = \frac{3}{4}$$

$$\therefore \int_{-1}^{1} f(x)\, dx = \frac{3}{4} + \frac{3}{4} = \frac{6}{4} = 1.5$$

$$\therefore\ f(x) = \int_{-1}^{1} \frac{1}{1+x^2}\ dx = 1.5\,.$$

By three point Gaussian formula, $\int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$

$$f(0) = \frac{1}{1+0} = 1$$

$$\therefore\ f\left(-\sqrt{\frac{3}{5}}\right) = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$$

$$\therefore\ f\left(\sqrt{\frac{3}{5}}\right) = \frac{1}{1+\frac{3}{5}} = \frac{5}{8}$$

$$\therefore \int_{-1}^{1} f(x)\, dx = \frac{8}{9} \times 1 + \frac{5}{9}\left\{\frac{5}{8} + \frac{5}{8}\right\}$$

$$= \frac{8}{9} + \frac{5}{9} \times \frac{10}{8}$$

$$\therefore \int_{-1}^{1} f(x)\, dx = 1.5833\,.$$

But exact value is

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 2\int_{0}^{1} \frac{dx}{1+x^2}$$

$$= 2\left[\tan^{-1}(x)\right]_0^1$$

$$= 2\left(\tan^{-1}(1) - \tan^{-1}(0)\right)$$

$$= 2\left(\frac{\pi}{4} - 0\right)$$

$$= \frac{\pi}{2}$$

$$\int_{-1}^{1} \frac{dx}{1+x^2} = 1.5708\,.$$

Thus, exact value is 1.5708.

By Gaussian two point formula value is 1.5.

By Gaussian three point formula value is 1.5833.

Example: 2

Using three point Gaussian Quadrature formula, evaluate $\int_{-1}^{1} \frac{x^2}{1+x^4}\, dx$.

Solution

Three point Gaussian Quadrature formula is $\int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$

Given $\int_{-1}^{1} \frac{x^2}{1+x^4}\, dx$

Here $f(x) = \frac{x^2}{1+x^4}$

$\therefore f(0) = \frac{0}{1+0} = 0$

$$f\left(-\sqrt{\frac{3}{5}}\right) = \frac{\frac{3}{5}}{1+\frac{3^2}{5^2}} = \frac{15}{34}$$

Similarly,

$$f\left(\sqrt{\frac{3}{5}}\right) = \frac{15}{34}$$

$$\int_{-1}^{1} f(x)dx = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right]$$

$$= \frac{8}{9}\times 0 + \frac{5}{9}\left[\frac{15}{34} + \frac{15}{34}\right]$$

$$\therefore \int_{-1}^{1} \frac{x^2}{1+x^4}\, dx = 0.4902.$$

Double integrals using Trapezoidal and Simpson's Rules

We shall evaluate double integral $I = \int_a^b \int_c^d f(x, y)\, dx\, dy$ using Trapezoidal and Simpson's rule.

The formula for the evaluation of a double integral can be obtained by repeatedly applying the Trapezoidal and Simpson's rules.

Trapezoidal Rule for Double Integral

The formula for the evaluation of double integral using Trapezoidal rule is,

$$I = \frac{hk}{4}\begin{bmatrix} (sum\ of\ the\ values\ at\ four\ corner\ of\ the\ box) + \\ 2(sum\ of\ the\ values\ at\ the\ boundary\ of\ the\ box\ except\ the\ corner) + \\ 4(sum\ of\ the\ remaing\ values) \end{bmatrix}$$

where h – length of x values and k – length of y values.

Simpson's Rule for Double Integral

The formula for the evaluation of double integral using Simpson's rule is,

$$I = \frac{hk}{9}\begin{bmatrix} (sum\ of\ the\ values\ at\ four\ corner\ of\ the\ box) + \\ 4(sum\ of\ the\ values\ at\ the\ boundary\ of\ the\ box\ except\ the\ corner) + \\ 16(sum\ of\ the\ remaing\ values) \end{bmatrix}$$

where h – length of x values and k – length of y values.

Example: 1

Evaluate $\int_0^2\int_0^2 f(x,y)\, dx\, dy$ by Trapezoidal rule for the following data:

y \ x	0	0.5	1	1.5	2
0	2	3	4	5	5
1	3	4	6	9	11
2	4	6	8	11	14

Solution

Here, h = 0.5, k = 1.

$$\therefore I = \int_0^2\int_0^2 f(x,y)\, dx\ dy$$

By Trapezoidal rule,

$$I = \frac{hk}{4}\begin{bmatrix}(sum\ of\ the\ values\ at\ four\ corner\ of\ the\ box)+ \\ 2(sum\ of\ the\ values\ at\ the\ boundary\ of\ the\ box\ except\ the\ corner)+ \\ 4(sum\ of\ the\ remaing\ values)\end{bmatrix}$$

$$= \frac{(0.5)\times 1}{4}[(2+5+14+4)+2(3+4+5+11+11+8+6+3)+4(4+6+9)]$$

I = 25.375.

Example: 2

Using Simpson's $\frac{1}{3}$ rule evaluate $\int_0^1\int_0^1 \frac{1}{1+x+y}\ dx\ dy$ taking h = k = 0.5.

Solution

x = 0, 0.5, 1; y = 0, 0.5, 1. The table values are

y \ x	0	0.5	1
0	1	0.6667	0.5
0.5	0.6667	0.5	0.4
1	0.5	0.4	0.3333

Let

$$g(y) = \int_0^1 \frac{1}{1+x+y}\ dx\ dy$$

By Simpson's rule,

$$I = \frac{hk}{9}\begin{bmatrix}(sum\ of\ the\ values\ at\ four\ corner\ of\ the\ box)+ \\ 4(sum\ of\ the\ values\ at\ the\ boundary\ of\ the\ box\ except\ the\ corner)+ \\ 16(sum\ of\ the\ remaing\ values)\end{bmatrix}$$

$$g(0) = \frac{0.5}{3}[(1+0.5)+4(0.6667)]+[1.5+2.66668]$$

$$g(0) = 0.69441889.$$

$$g(0.5) = \frac{0.5}{3}[(0.6667+0.4)+[3.0667+4(0.5)]]$$

$g(0.5) = 0.51111.$

$g(1) = \frac{0.5}{3}[(0.5 + 0.3333) + 4(0.4)]$

$g(1) = 0.405538.$

Hence

$I = \int_0^1 g(y)\, dy$

$= \frac{0.5}{3}[(0.9441889 + 0.405538) + 4(0.51111)]$

$I \cong 0.5241.$

3.7 Weddle's Rule

Put n = 6 in Newton-Cot's Quadratic formula and neglecting all differences of orders higher than sixth, we get

$$\int_{x_0}^{x_{0+nh}} f(x)dx = \frac{3h}{10}[(y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6) + (y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12})$$

$$+ \cdots + (y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n)]$$

This equation is called Weddle's rule.

$x_5 = x_0 + 5h$

Example: 1

Evaluate $\int_0^1 \frac{dx}{1+x}$ using Weddle's rule with 6 equal intervals.

Solution

Here n = 6, $\therefore h = \frac{1}{6}$,

Let $y = f(x) = \frac{1}{1+x}$ $\therefore x = 0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \cdots, \frac{6}{6}$.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y = \frac{1}{1+x}$	1	0.8571	0.75	0.667	0.6	0.5455	0.5

By Weddle's rule,

$$\int_0^1 \frac{1}{1+x}dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5]$$

$$= \frac{3\left(\frac{1}{6}\right)}{10}[1 + 5(0.8571) + 0.75 + 6(0.6667) + 0.6 + 5(0.5455 + 0.5]$$

= 0.69320.

3.8 Newton's – Cote's Formula (or) Cote's Formula

Newton's-Cote's formula gives a way for computing the integral $\int_a^b f(x)dx$ numerically, when y = f(x) is known at equidistance values of x, but its derivation is based on the integration of Lagrange's interpolation formula.

$$\int_a^b f(x)dx = h\sum_{k=0}^{n} y_k \int_0^n C_k(u)du$$

Where $C_k(u) = (-1)^{n-k}\frac{[u]^k.[-n+u]^{n-k}}{k!(n-k)!}$

Here $C_k(u)$ is a polynomial of degree n equation is called the cote's polynomial.

Example: 1

Find the Cote's polynomials for n=1.

Solution

The Cote's polynomials are $C_0(u)$ and $C_1(u)$

$$C_k(u) = (-1)^{n-k}\frac{[u]^k.[-n+u]^{n-k}}{k!(n-k)!}$$

Put n = 1 and k = 0 in $C_k(u)$

$$C_0(u) = \frac{[u]^0.[-1+u]^1}{0!(1-0)!} = -u + 1.$$

$$C_1(u) = \frac{[u]^1.[-1+u]^0}{1!(1-1)!} = u.$$

Example: 2

Find the Cote's polynomials for n=2.

Solution

$$C_k(u) = (-1)^{n-k}\frac{[u]^k.[-n+u]^{n-k}}{k!(n-k)!}$$

$$C_0(u) = \frac{[u]^0.[-2+u]^2}{0!(2-0)!}$$

$$= \frac{(-2+u)(-1+u)}{2}$$

$$= \frac{2-3u+u^2}{2}$$

$$C_1(u) = \frac{[u]^1.[-2+u]^1}{1!(2-1)!}$$

$$= \frac{u(-2+u)}{1}$$

$= 2u-u^2$

$$C_2(u) = \frac{[u]^2.[-2+u]^0}{2!(2-2)!}$$

$$= \frac{u(u-1)}{2}$$

$$C_2(u) = \frac{-u+u^2}{2}.$$

Exercises:

1. Evaluate the integral $\int_1^2 \frac{dx}{1+x^2}$ using Trapezoidal rule with two subintervals.
2. Dividing the range into 10 equal parts, find the value of $\int_0^{\frac{\pi}{2}} \sin x \, dx$ by (i) Trapezoidal rule (ii) Simpson's rule.
3. Using Simpson's one third rule evaluate $\int_0^1 xe^x \, dx$ taking 4 intervals. Compare your result with actual value.
4. Calculate $\int_{0.5}^{0.7} e^{-x}\sqrt{x} \, dx$ taking 5 ordinates by Simpson's rule.
5. Evaluate $\int_0^2 \frac{dx}{x^2+4}$ using Romberg's method. Hence, obtain an approximate value of π.
6. Using Romberg's method, Evaluate $\int_0^{\pi} \sin x \, dx$ correct to four decimal places.
7. Using three point Gaussian Quadrature formula, Evaluate $\int_0^1 \frac{1}{1+t^2} \, dt$.

8. Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian three point formula.

9. Evaluate $\int_{0.2}^{1.5} e^{-x^2}\,dx$ using the three point Gaussian Quadrature.

10. Evaluate $\int_0^1 \frac{dx}{1+x}$ by Gaussian formula with two points.

11. Use Gaussian three point formula and evaluate $\int_1^5 \frac{1}{x}\,dx$.

12. Using Gaussian three point formula, evaluate (i) $\int_{-1}^{1}\left(3x^2+5x^4\right)dx$ (ii) $\int_0^1\left(3x^2+5x^4\right)dx$. Also compare with exact values.

13. Evaluate $\int_{-2}^{2} e^{-\frac{x}{2}}\,dx$ by Gaussian two point formula.

14. Evaluate $\int_0^1\int_1^2 \frac{2xy}{(1+x^2)(1+y^2)}\,dx\,dy$ by Trapezoidal rule with h = k = 0.25.

15. Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x^2+y^2}$ numerically with h = 0.2 along x direction and k = 0.25 along y direction.

16. The function f(x, y) is defined by the following table. Compute $\int_1^3\int_0^2 f(x,y)\,dx\,dy$ using Simpson's rule in both direction.

y \ x	0	0.5	1	1.5	2
1	2	1.5	1.3	1.4	1.6
2	3.1	2.5	2	2.3	2.9
3	4.2	4	3.8	4.1	4.4

17. Apply Weddle's rule to evaluate the approximate value of the integral $\int_0^6 \frac{\mathrm{dx}}{1+\mathrm{x}^2}$ by dividing the range into 6 equal parts.

18. Compute Cote's polynomials for n = 3, 4, 5 and 6.

19. Compute Cote Numbers for n=1, 2, 3, 4, 5 and 6.

20. Verify that sum of Cote's Numbers is 1 for n=1, 2, 3, 4, 5 and 6.

*****

UNIT-IV

NUMERICAL SOLUTIONS OF DIFFERENTIAL EQUATIONS

4.1 Introduction

Many problems in science and engineering can be reduced to the problem of solving differential equations satisfying given conditions. By applying analytical methods we can solve several standard types of differential equations. However the differential equations appearing in physical problems are quite complex and may not posses closed form solutions. In such cases they can be solved numerically.

We know that the general solution of a differential equation of the n^{th} order has n arbitrary constants. In order to compute the numerical solution of such an equation we need n conditions. If all the n conditions are specified at the initial point only then it is called an initial value problem. If the conditions are specified at two or more points, then it is called a boundary value problem.

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$.

This problem can be solved any of the methods give the solution in one of the two forms given below:

(i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution. The methods of Taylor and Picard belong to this type. In these methods y is approximated by a truncated series and each term of the series is a function of x. The information about the curve at one point is used and the solution is not iterated. Hence these methods are called single step methods or point wise methods. A solution of this type is called a point wise solution.

(ii) Given a set of tabulated values of x and y, we obtain y by iterative process. The methods of Euler, Runge – Kutta, Milne, Adams – Bashforth etc. belong to this type. In these methods, the values of y are computed by short steps ahead for equal intervals h of the independent variable. These values are iterated till we get the desired accuracy. Hence these methods are called step by step methods.

4.2 Picard's Method

Consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad (1)$$

with initial condition $y = y_0$ when $x = x_0$.

We now replace equation (1) by an equivalent integral equation.

Integrating equation (1) we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$$

$$\text{(i.e)} \quad y = y_0 + \int_{x_0}^{x} f(x, y)dx \qquad (2)$$

This is an integral equation which contains the unknown y under the integral sign. Equation (2) is equivalent to equation (1) since any solution of equation (2) is a solution of equation (1) and vice versa.

The first approximation y_1 to the solution is obtained by putting $y = y_0$ in f(x,y) and from equation (2) we have

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx .$$

Similarly for the second approximation y_2, put $y = y_1$ in f(x,y) and from equation (2) we have

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx .$$

Continuing this process the n^{th} approximation is given by

$$y_n = y_0 + \int_{x_0}^{x} f\left(x, y_{n-1}\right)dx .$$

This is known as Picard's iteration formula.

Note:

Picard's method gives a sequence of approximations y_1, y_2, ... each giving a better result than the preceding one. But this can be applied only to equations in which the successive integration can be obtained easily.

Example: 1

Using Picard's method solve $\frac{dy}{dx} = 1 + xy$ with y(0) = 2. Find y(0.1), y(0.2) and y(0.3).

Solution

The Picard's iteration formula for the differential equation $\frac{dy}{dx} = f\left(x, y\right)$ is $y_n = y_0 + \int_{x_0}^{x} f\left(x, y_{n-1}\right)dx$ where n = 1, 2, ...

Given f(x, y) = 1+ xy, $x_0 = 0$ and $y_0 = 2$.

The first approximation is

$$y_1 = y_0 + \int_{x_0}^{x} f\left(x, y\right)dx$$

$$= 2 + \int_{0}^{x} f\left(x, 2\right)dx$$

$$= 2 + \int_{0}^{x} \left(1 + 2x\right)dx$$

$$y_1 = 2 + x + x^2 .$$

The second approximation is

$$y_2 = y_0 + \int_{x_0}^{x} f\left(x, y_1\right)dx$$

$$= 2 + \int_{0}^{x} \left[1 + x\left(2 + x + x^2\right)\right]dx$$

$$y_2 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} .$$

The third approximation is

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$= 2 + \int_{x_0}^{x} \left[1 + x\left(2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4}\right)\right]dx$$

$$y_3 = 2 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} + \frac{x^6}{24}.$$

Putting x = 0.1, 0.2 and 0.3 in equation (1) we get

$$y_1 = y(0.1) = 2.1104$$

$$y_2 = y(0.2) = 2.2431$$

$$y_3 = y(0.3) = 2.4012 \cdot$$

Example: 2

Find the value of y(0.1) by Picard's method given $\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0)=1$.

Solution

The Picard's iterative formula for the differential equation $\frac{dy}{dx} = f(x,y)$ is $y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$ where $n = 1,2,...$

Here $f(x,y) = \frac{y-x}{x+y}, x_0 = 0$ and $y_0 = 1$.

The first approximation is

$$y_1 = y_0 + \int_0^x f(x, y_0)dx$$

$$= 1 + \int_0^x \left(\frac{1-x}{1+x}\right)dx$$

$$= 1 + \int_0^x \left(-1 + \frac{2}{1+x}\right)dx \text{ (By partial fraction)}$$

$$= 1 + \left[-x + 2\log_e(1+x)\right]_0^x$$

$$y_1 = 1 - x + 2\log_e(1+x).$$

Putting x = 0.1 we get,

$$y_1 = y(0.1) = 1 - 0.1 + 2\log_e(1+0.1)$$

$$= 0.9 + 2 \times 0.0953$$

$$y_1 = 1.0906.$$

4.3 Initial Value Problem for Ordinary Difference Equations

The differential equation together with initial conditions is called an initial value problem. In this unit, we are going to solve numerically, the first order initial value problem defined by,

$$\frac{dy}{dx} = f(x, y); y(x_0) = y_0 .$$

The solution of such initial value problem can be obtained by two different methods:

1. Single step method
2. Multi step method.

The following are the single step method:

1. Euler method
2. Modified Euler method
3. Taylor series method
4. Runge - Kutta method.

All the above methods, require the information at a single point at $x = x_0$.

The following are the multi step methods:

1. Milne's method
2. Adam's – Bashforth method.

Euler and Modified Euler methods

Taylor's series method and Picard's method are used to yield the solution of a differential equation in the form of power series. But Euler methods are used to find the solution in the form of table values of equally spaced points.

Euler Method

The formula is $y_{n+1} = y_n + hf(x_n, y_n); n = 0,1,2,3,\cdots$

Modified Euler method Formula

$$y_{n+1} = y_n + hf\left[x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right].$$

Example: 1

Using Euler's method, find y(0.2), y(0.4), y(0.6) from $\frac{dy}{dx} = x + y; y(0) = 1$.

Solution

x	0	0.2	0.4	0.6
y	1	y_1	y_2	y_3

Given $y' = x + y$; h=$x_1 - x_0$

Here h = 0.2

$$y' = \frac{dy}{dx} = f(x, y) = x + y$$

By Euler method

To find y_1 = y(0.2)

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.2(x_0 + y_0)$$

= 1 + 0.2 (0+1) = 1.2

y_1 = 1.2

To find y_1 = y(0.4)

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.2 + 0.2(x_1 + y_1)$$

= 1.2 + 0.2 (0.2+1.2) = 1.48

y_2 = 1.48

To find y_3 = y(0.6)

$$y_3 = y_2 + hf(x_2, y_2)$$

$$= 1.48 + 0.2(x_2 + y_2)$$

= 1.48 + 0.2 (0.4+1.48) = 1.48

y_3 = 1.856.

Example: 2

Using Euler's method, solve $y' = x + y + xy$, y(0) =1. Compute y at x = 0.1, by taking h = 0.05.

Solution

x	0	0.05	0.10
y	1	y_1	y_2

Given $y' = x + y + xy = f(x, y)$; h=0.05

$$\frac{dy}{dx} = f(x, y) = x + y + xy$$

By Euler method

To find $y_1 = y(0.05)$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 1 + 0.05(x_0 + y_0 + x_0y_0)$$

$$= 1 + 0.05\,(0+1+0) = 1.05$$

$y_1 = 1.05$

To find $y_2 = y(0.10)$

$y_2 = y_1 + hf\,(x_0, y_0)$

$$= 1.05 + 0.05(x_1 + y_1 + x_1y_1)$$

$$= 1.05 + 0.05\,(0.05+1.05+0.05\times1.05) = 1.107625$$

$y_2 = 1.107625.$

Example: 3

Compute y at x = 0.25 by modified Euler method given $y' = 2xy$, $y(0) = 1$

Solution

Given $y' = 2xy = f(x, y)$ $\frac{dy}{dx} = 2xy$

x	0	0.25
y	1	y_1

To find $y_1 = y(0.25)$

By modified Euler method,

$$y_{n+1} = y_n + hf\left[x_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(x_n, y_n)\right]$$

$$y_1 = y_0 + hf\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2}f(x_0, y_0)\right]$$

$$y_1 = 1 + 0.25f\left[0 + \frac{0.25}{2}, 1 + \frac{0.25}{2}.2x_0y_0\right]$$

$y_1 = y(0.25) = 1.625.$

Example: 4

Using modified Euler's method, compute y(0.1) with h=0.1 from $y' = y - \frac{2x}{y}, y(0) = 1.$

Solution

x	0	0.1
y	1	y_1

Given $y' = \frac{dy}{dx} = f(x, y) = y - \frac{2x}{y}$

To find $y_1 = y(0.1)$

By modified Euler method,

$$y_{n+1} = y_n + h f\left[x_n + \frac{1}{2}h, y_n + \frac{1}{2}h f(x_n, y_n)\right]$$

$$y_1 = y_0 + h f\left[x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right]$$

$$y_1 = 1 + 0.1 f\left[0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}.\left(y_0 - \frac{2x_0}{y_0}\right)\right]$$

$y_1 = y(0.1) = 1.09548.$

Taylor Series Method

Consider the differential equation

$$y' = \frac{dy}{dx} = f(x, y); y(x_0) = y_0$$

The solution of the above equation obtained by Taylor series as follows:

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y'_0 + \frac{(x - x_0)^2}{2!} y''_0 + \frac{(x - x_0)^3}{3!} y'''_0 + \cdots$$

It is called power series solution.

In general, the Taylor's algorithm is given as follows:

$y_{n+1} = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \cdots$ where n = 0, 1, 2, 3, …

where h is the step size; $h = x_1 - x_0$.

Example: 1

Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2, correct to four decimal places from $\frac{dy}{dx} = x^2 y - 1, y(0) = 1$.

Solution

Here $x_0 = 0$; $y_0 = 1$

$\therefore y' = x^2 y - 1$ $\quad\quad y'_0 = x_0^2 y_0 - 1$

$= 0 (1) -1 = -1$

$y' = x^2 y - 1$	$y'_0 = -1$

$y'' = x^2y' + y2x$ $y'' = x^2y' + 2xy$	$y''_0 = x_0^2y'_0 + 2x_0y_0$ $y''_0 = 0$
$y''' = 2xy' + y2 + x^2y'' + y'2x$ $y''' = 2y + 4xy' + x^2y''$	$y'''_0 = 2y_0 + 4x_0y'_0 + x'_0y''_0$ $y'''_0 = 2$
$y^{(iv)} = 2y' + 4(xy'' + y') + (x^2y''' + y''2x)$ $y^{(iv)} = 6y' + 6xy'' + x^2y'''$	$y^{(iv)} = 2y'_0 + 4(x_0y''_0 + y'_0)$ $y^{(iv)} = -6$

Therefore Taylor series of y(x) about $x_0 = 0$ is given by

$$y(x) = y_0 + \frac{(x - x_0)}{1!}y'_0 + \frac{(x - x_0)^2}{2!}y''_0 + \frac{(x - x_0)^3}{3!}y'''_0 + \cdots$$

$$y(x) = 1 + \frac{(x - 0)}{1!}(-1) + \frac{(x - 0)^2}{2!}\times 0 + \frac{(x - 0)^3}{3!}(2) + \frac{(x - 0)^4}{4!}(-6) + \cdots$$

$$y(x) = 1 - \frac{x}{1!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \ldots$$

Hence y(0.1) = 0.9003

Also, y(0.2) = 0.8023.

Example: 2

Solve $y' = y^2 + x$; y(0)=1 using Taylor series method and computer y(0.1) and y(0.2).

Solution

Given y(0) = 1

Here $x_0 = 0$; $y_0 = 1$.

$y' = x^2 + x$	$y'_0 = 1$
$y'' = 2yy' + 1$	$y''_0 = 3$
$y''' = 2(yy'' + y'y') + 4xy' + x^2y''$	$y'''_0 = 8$
$y^{(iv)} = 2(yy''' + y''y' + y'y'' + y'y'')$	$y^{(iv)} = 34$

To find y(0.1):

Let $y_1 = y(x_1)$

Taylor algorithm is $y_1 = y_0 + \frac{h}{1!}y'_0 + \frac{h^2}{2!}y''_0 + \frac{h^3}{3!}y'''_0 + \cdots$

$$y(0.1) = 1 + \frac{0.1}{1!}(1) + \frac{0.01}{2!}(3) + \frac{0.001}{3!}(8) + \frac{0.0001}{4!}(34) + \cdots$$

$$y(0.1) = 1.116411$$

To find y(0.2):

Let $y_2 = y(x_2)$

Then by Taylor's algorithm

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \cdots \text{ where } h = 0.1$$

Given $y' = y^2 + x$

$$y_1' = {y_1}^2 + x_1 = 1.3463$$

$$y_1'' = 2y_1 2y_1' = 4.006$$

$$y_1''' = 2yy'' + 2y'^2 = 12.5696$$

$$y(0.2) = 1.1164 + (0.1)(1.3463) + \frac{0.01}{2}(4.006) + \frac{0.001}{3!}(12.5696) + \cdots$$

y(0.2) = 1.2732.

4.4 Multistep Method (Predictor-corrector method)

Predictor-corrector methods are methods which require function values at x_n, x_{n-1}, x_{n-2}, x_{n-3}, for the computation of the function value at x_{n+1}.

A predictor is used to find the value of y at x_{n+1} and then a corrector formula is used to improve the value of y_{n+1}.

The following two methods are multi-step methods:

1. Milne's predictor-corrector method
2. Adam's Predictor-corrector method.

Milne's predictor-corrector method

(i) To use Milne's predictor-corrector method, we need at least 4 values prior to the required values.

(ii) Knowing four consecutive values of y namely y_{n-3}, y_{n-2}, y_{n-1}, y_n, we calculate y_{n+1} using predictor formula. (Use y_{n+1} on right hand formula to get, better y_{n+1} after correction)

(iii) Predictor formula is used to predict the values of y_{n+1} at x_{n+1} and then a corrector formula is used to improve the values of y_{n+1}.

(iv) In Milne's predictor corrector method may have been computed by Taylor's series or Euler's method of modified Euler's method or Runge - Kutta method or Picard's method.

Milne's Predictor Formula:

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right].$$

Milne's Corrector Formula

$y_{4,c} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$ where $y'_4 = f(x_4, y_4, p)$.

Example: 1

Using Milne's method, compute y(0.8) given that $\frac{dy}{dx} = 1 + y^2$ with y(0) = 1, y(0.2) = 0.2027, y(0.4) = 0.4228 and y(0.6) = 0.6841.

Solution

Given

	x	y	$y' = 1 + y^2$
x_0	0	(y_0)	$y'_0 = 1 + y_0^2 = 1 + 1^2 = 2$
x_1	0.2	0.2027	$y'_1 = 1 + y_1^2 = 1 + (0.2027)^2 = 1.0410$
x_2	0.4	0.4228	$y'_2 = 1 + y_2^2 = 1 + (0.4228)^2 = 1.1787$
x_3	0.6	0.684	$y'_3 = 1 + y_3^2 = 1 + (0.6841) = 1.4681$
x_4	0.8	?	?

To find y(0.8):

$x_4 = 0.8$; $h = 0.2$

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}[2y'_1 - y'_2 + 2y'_3]$$

$$= 1.0239$$

$y'_4 = f(x_4, y_4) = 1 + (1.02398)^2 = 2.0480$

$y'_4 = 2.0480.$

By Milne's Corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}[y'_2 - 4y'_3 + y'_4]$$

$$= 0.4228 + \frac{0.2}{3}[1.178 + 4(1.4681) + 2.0480]$$

y(0.8) = 1.0294.

Example: 2

Given $\frac{dy}{dx} = x^3 + y$, y(0) = 2, the values of y(0.2) = 2.073, y(0.4) = 2.452 and y(0.6) = 3.023 are got by Runge - Kutta method of fourth order, Find y(0.8) by Milne's predictor -corrector method taking h = 0.2.

Solution

Given

	x	y	$y' = x^3 + y$
x_0	0	$2y_0$	$y'_0 = x_0^3 + y_0$ =0+2 = 2
x_1	0.2	2.073	$y'_1 = x_1^3 + y_1$ =$(0.2)^3$ +2.073 = 2.081
x_2	0.4	2.452	$y'_2 = x_2^3 + y_2$ =$(0.4)^3$ +2.452 = 2.516
x_3	0.6	3.023	$y'_3 = x_3^3 + y_3$ = $(0.6)^3$ + 3.023 = 3.239
x_4	0.8	?	?

To find y(0.8):

x_4 = 0.8; h=0.2

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y'_1 - y'_2 + 2y'_3\right]$$

$$= 2 + \frac{4(0.2)}{3}\left[2(2.081) - 2.516 + 2(3.239)\right]$$

= 4.1664

$$\therefore y'_4 = f(x_4, y_{4,p})$$

= f(0.8, 4.1664)

= $(0.8)^3$ + 4.1664 = 4.6784

$y'_4 = 4.6784$

By Milne's Corrector Formula,

$$y_{4,c} = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$

$$= 2.452 + \frac{0.2}{3}\left[2.516 + 4(3.239) + 4.6784\right]$$

= 3.79536

The corrected value of y(0.8) = 3.79536.

4.5 Runge - Kutta method

The Taylor's series method of solving differential equations numerically in restricted because of the evaluation of the higher order derivatives Runge - Kutta methods of solving initial value problems do not require the calculations of higher order derivatives and give greater accuracy.

Second – Order Runge - Kutta method

Consider $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$, then the value of y_1 is obtained as follows:

$\therefore y_1 = y_0 + \Delta y$ where $\Delta y = k_2$

$$\text{where } k_2 = hf\left[x + \frac{h}{2}, y + \frac{k_1}{2}\right]$$

$$k_1 = hf(x, y).$$

Third – Order Runge - Kutta method

Consider $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$, then the value of y is obtained as follows:

$$y_1 = y_0 + \Delta y$$

$$\text{where } \Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

$$\text{where } k_1 = hf(x, y)$$

$$k_2 = hf\left(x + \frac{h}{2}, y + \frac{k_1}{2}\right)$$

$$k_3 = hf(x + h, y + 2k_2 - k_1).$$

Fourth – Order Runge - Kutta method

This method is commonly used for solving the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$.

The value of $y_1 = y(x_1)$ is obtained as follows:

To find y_1

$$y_1 = y_0 + \Delta y$$

$$\text{Where } \Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$\text{where } k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3).$$

Example: 1

Given $\frac{dy}{dx} = x^3 + y$, y(0) = 2. Compute y(0.2), y(0.4) and y(0.6) by Runge - Kutta method of fourth order.

Solution

Given $y' = f(x, y) = x^3 + y$

Also

x	0	0.2	0.4	0.6
y	2	y_1	y_2	y_3

To find y_1:

Fourth order Runge - Kutta formula is

$$y_1 = y_0 + \Delta y$$

Where $\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_1 = 0.2(0,2) = 0.2(0^3 + 2) = 0.4$$

$$k_2 = 0.2\left(0 + \frac{0.2}{2}, 2 + \frac{0.4}{2}\right) = 0.4402$$

$$k_3 = 0.2\left(0 + \frac{0.2}{2}, 2 + \frac{0.4402}{2}\right) = 0.44422$$

$$k_4 = 0.2(0 + 0.2, 2 + 0.44422) = 0.490444$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.4 + 2(0.4402) + 2(0.44422) + 0.490444] = 0.44321$$

$y_1 = y_0 + \Delta y = 2 + 0.44321 = 2.44321$

$y(0.2) \approx 2.44321$.

Similarly, $y_2 = y(0.4) = 2.99$ ($k_1 = 6.4902$, $k_2 = 0.5430$, $k_3 = 0.5483$, $k4 = 0.6111$, $\Delta y = 0.5473$)

$y_3 = y(0.6) = 3.68$ ($k_1 = 0.6108$, $k_2 = 0.6841$, $k_3 = 0.6914$, $k4 = 0.7795$, $\Delta y = 0.6902$).

Example: 2

Using Runge - Kutta method of 4^{th} order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x}$ with $y(0) = 1$ at $x = 0.2$.

Solution

Given $y' = \frac{y^2 - x^2}{y^2 + x}$; $h = 0.2$

x	0	0.2
y	1	y_1

To find y_1:

Fourth order Runge - Kutta formula is

$$y_1 = y_0 + \Delta y$$

where $\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

where $k_1 = hf(x_0, y_0)$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf\left(x_0 + h, y_0 + k_3\right)$$

$$k_1 = 0.2\left[\frac{y^2 - x^2}{y^2 + x}\right] = 0.2$$

$$k_2 = 0.2f\left(0 + \frac{0.2}{2}, 1 + \frac{0.2}{2}\right) = 0.19672$$

$$k_3 = 0.1967$$

$$k_4 = 0.1891$$

$y_1 = y(0.2) = y_0 + \Delta y = 1 + 0.19598 = 1.19598$.

4.6 Solution of Algebraic and Transcendent Equations

In mathematics, we often come across problems of obtaining solutions of equations of the form $f(x) = 0$. If $f(x)$ is a polynomial then the equation $f(x) = 0$ is called an algebraic equation.

Equations which involve transcendental functions like sin x, cos x, tan x, log x and e^x etc., are called transcendental equations.

$x^2 + 5x + 6 = 0,\ 2x^2 - x + 4 = 0, x^5 - x^3 + 2x + 3 = 0$ and some examples of algebraic equations.

$3x + \sin x + 2 = 0,\ \log_{10} x - 2x = 10,\ a\,e^x + b\,\sin x + c\,\cos\,x + d\,\log x + x = 0$ are some examples of transcendental equations.

If f(x) = 0 is a quadratic equation $ax^2 + bx + x = 0$, we have a simple formula namely $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to find its roots.

However, if f(x) is a polynomial of higher degree or an expression involving transcendental functions we have no simple formula to find roots.

Due to limitations of analytical methods, formula giving exact numerical values of the solutions exist any in very simple cases.

Hence, we have to use approximate methods to get solutions with good degree of accuracy.

We have different methods for obtaining approximate solutions for algebraic and transcendental equations.

(i) Iterative Method

(ii) Aitken's Δ^2 Method

(iii) Bisection Method

(iv) Regula – Falsi Method

(v) Newton – Raphson Method.

Iterative Method (or) Method of Successive Approximation (or) fixed Point Method

To solve the equation f(x) = 0 by the iteration method, we start with the approximation value of the root. The equation f(x) = 0 is expressed as $x = \phi(x)$ is called fixed point equation.

If x_0 is the starting approximate value to the actual root 'α' of $x = \phi(x)$, be first approximation is $x_1 = \phi(x_0)$, second approximation is $x_2 = \phi(x_1)$ and so on.

In general we have $x_n = \phi(x_{n-1})$, n = 1, 2, 3, ... Here x_n is the n^{th} iteration and the values of x_n gives the root of the given equation at the n^{th} iteration.

Sufficient Condition for Convergence of Iteration (statement of fixed point Theorem)

Let $x = \alpha$ be a root of the equation f(x) = 0 which is equivalent to $x = \phi(x)$. Let I be any interval containing the root α. If $|\phi'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, \cdots, x_n$ to the root α, provided the initial approximation x_0 is chosen in I.

Example: 1

Solve the equation $x^3 + x^2 - 1 = 0$ for the root by iteration method correct to 4 decimal places.

Solution

Let $f(x) = x^3 + x^2 - 1$

f(0) = -1 (negative)

f(1) = 1 (positive)

The root lies between 0 and 1.

Let $x_0 = 0.5$.

Express f(x) = 0 as x = ϕ(x).

$x^3 + x^2 - 1 = 0$

$\Rightarrow x^3 + x^2 = 1$

$\Rightarrow x^2 (x+1) = 1$

$$\Rightarrow x = \frac{1}{x+1}$$

$$\Rightarrow x = \frac{1}{\sqrt{x+1}} = (1+x)^{-\frac{1}{2}}$$

$$\therefore x = \phi(x) = (1+x)^{-\frac{1}{2}}$$

$$\phi'(x) = -\frac{1}{2}(1+x)^{-\frac{3}{2}}$$

$$|\phi'(x)| = \frac{1}{2(1+x)^{\frac{3}{2}}}$$

Here I = [0, 1].

$$\therefore |\phi'(0)| = \frac{1}{2} = 0.5 < 1$$

$$\therefore |\phi'(1)| = \frac{1}{2 \times 2^{\frac{3}{2}}} = 0.1768 < 1.$$

The condition of convergence is satisfied.

Iteration formula:

$x_n = \phi(x_{n-1})$; n = 1, 2, 3, ...

Iteration 1: n = 1; x_0 = initial value is 0.5

$$\therefore x_1 = \phi(x_0) = \frac{1}{\sqrt{1+x_0}} = \frac{1}{\sqrt{1+0.5}} = 0.8165.$$

Iteration 2: n = 2,

$$\therefore x_2 = \phi(x_1) = \frac{1}{\sqrt{1+x_1}} = \frac{1}{\sqrt{1+0.8165}} = 0.742.$$

Iteration 3: n = 3,

$$\therefore x_3 = \phi(x_2) = \frac{1}{\sqrt{1+x_2}} = \frac{1}{\sqrt{1+0.742}} = 0.7577.$$

Iteration 4: n = 4,

$$\therefore x_4 = \phi(x_3) = \frac{1}{\sqrt{1+x_3}} = \frac{1}{\sqrt{1+0.7577}} = 0.7543.$$

Iteration 5: n = 5,

$$\therefore x_5 = \phi(x_4) = \frac{1}{\sqrt{1+x_4}} = \frac{1}{\sqrt{1+0.7543}} = 0.7550.$$

Iteration 6: n = 6,

$$\therefore x_6 = \phi(x_6) = \frac{1}{\sqrt{1+x_5}} = \frac{1}{\sqrt{1+0.7550}} = 0.7549.$$

Iteration 7: n = 7,

$$\therefore x_7 = \phi(x_6) = \frac{1}{\sqrt{1+x_6}} = \frac{1}{\sqrt{1+0.7549}} = 0.7549.$$

From 6^{th} and 7^{th} iterations, x = 0.7549.

Example 2:

Find the cube root of 15 correct to 4 decimal places by the iteration method.

Solution

Let x be the cube root of 15.

$X^3 = 15$.

$f(x) = x^3 - 15$

The equation is $x^3 - 15 = 0$

$\Rightarrow x^3 = 15$

$$\Rightarrow x = (15)^{\frac{1}{3}}$$

$\Rightarrow x \neq \phi(x)$, which is constant, convergence is not satisfied.

$$\therefore x^2 \times x = 15$$

$$\Rightarrow x^2 = \frac{15}{x} \Rightarrow x = \sqrt{\frac{15}{x}} \Rightarrow x = \phi(x).$$

$$\therefore \phi(x) = \sqrt{\frac{15}{x}} = 15\ x^{-\frac{1}{2}}.$$

$$\therefore \phi'(x) = \sqrt{15}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}.$$

$$\therefore |\phi'(x)| = \frac{\sqrt{15}}{2\ x^{\frac{3}{2}}}.$$

$f(x) = x^3 - 15$

$f(2) = 8 - 15 = -7$ (negative)

$f(3) = 27 – 15 = 12$ (positive)

The root lies between 2 and 3.

Let $x_0 = 2.5$; $\phi(x) = \sqrt{\frac{15}{x}}$.

Iteration formula is $x_n = \phi(x_{n-1})$; n = 1, 2, 3, ...

$$x_1 = \phi(x_0) = \phi(2.5) = \sqrt{\frac{15}{2.5}} = 2.4495.$$

$$x_2 = \phi(x_1) = \phi(2.4495) = \sqrt{\frac{15}{2.4495}} = 2.4746.$$

$$x_3 = \phi(x_2) = \phi(2.4746) = \sqrt{\frac{15}{2.4746}} = 2.4620.$$

$$x_4 = \phi(x_3) = \phi(2.4620) = \sqrt{\frac{15}{2.4620}} = 2.4683.$$

$$x_5 = \phi(x_4) = \phi(2.4683) = \sqrt{\frac{15}{2.4683}} = 2.4652.$$

$$x_6 = \phi(x_5) = \phi(2.4652) = \sqrt{\frac{15}{2.4652}} = 2.4667.$$

$$x_7 = \phi(x_6) = \phi(2.4667) = \sqrt{\frac{15}{2.4667}} = 2.4659.$$

$$x_8 = \phi(x_7) = \phi(2.4659) = \sqrt{\frac{15}{2.4659}} = 2.4663.$$

$$x_9 = \phi(x_8) = \phi(2.4663) = \sqrt{\frac{15}{2.4663}} = 2.4662.$$

$$x_{10} = \phi(x_9) = \phi(2.4662) = \sqrt{\frac{15}{2.4662}} = 2.4662.$$

$$\therefore x = 2.4662.$$

Newton – Raphson Method (or) Newton's Method (or) Method of Tangents

This method starts with an initial approximation to the root of the equation. A better and closer approximation to the root can be found by using an iterative process.

Newton – Raphson formula:

$$x_{i+1} = x_i - \left[\frac{f(x_i)}{f'(x_i)}\right]; \; i = 0,1,2,\cdots$$

Example: 1

Using Newton – Raphson method, find the root of $x^3 - 6x + 4 = 0$ and correct its 4 decimal points.

Solution

Newton – Raphson formula is

$$x_{i+1} = x_i - \left[\frac{f(x_i)}{f'(x_i)}\right]; \; i = 0,1,2,...$$

Let $f(x) = x^3 - 6x + 4; \;\; f'(x) = 3x^2 - 6$

$\therefore f(0) = 4$ (positive)

f(1) = -1 (negative)

The root lies between 0 and 1.

Let the initial approximation be $x_0 = 0.5$.

Iteration: 1, i = 0

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 0.5 - \left[\frac{(0.5)^3 - 6(0.5) + 4}{3(0.5)^2 - 6}\right] = 0.7143\,.$$

Iteration: 2, i = 1

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 0.7143 - \left[\frac{(0.7143)^3 - 6(0.7143) + 4}{3(0.7143)^2 - 6}\right] = 0.7319\,.$$

Iteration: 3, i = 2

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 0.7319 - \left[\frac{(0.7319)^3 - 6(0.7319) + 4}{3(0.7319)^2 - 6}\right] = 0.7320\,.$$

Iteration: 4, i = 3

$$x_4 = x_3 - \left[\frac{f(x_3)}{f'(x_3)}\right] = 0.7320 - \left[\frac{(0.7320)^3 - 6(0.7320) + 4}{3(0.7320)^2 - 6}\right] = 0.7320\,.$$

The value of x is 0.7320.

Example: 2

Solve by Newton - Raphson method $x^4 - x - 10 = 0$.

Solution

Let $f(x) = x^4 - x - 10; f'(x) = 4x^3 - 1$

f(0) = -10 (negative)

f(1) = -10 (negative)

f(2) = 5 (positive)

The root lies between 0 and 2.

Let the initial approximation be $x_0 = 1.5$.

Newton – Raphson formula is,

$$x_{i+1} = x_i - \left[\frac{f(x_i)}{f'(x_i)}\right]; i = 0,1,2,...$$

Iteration: 1, i = 0

$$x_1 = x_0 - \left[\frac{f(x_0)}{f'(x_0)}\right] = 1.5 - \left[\frac{(1.5)^4 - 1.5 - 10}{4(1.5)^3 - 1}\right] = 2.015.$$

Iteration: 2, i = 1

$$x_2 = x_1 - \left[\frac{f(x_1)}{f'(x_1)}\right] = 2.015 - \left[\frac{(2.015)^4 - 2.015 - 10}{4(2.015)^3 - 1}\right] = 1.8741.$$

Iteration: 3, i = 2

$$x_3 = x_2 - \left[\frac{f(x_2)}{f'(x_2)}\right] = 1.8741 - \left[\frac{(1.8741)^4 - 1.8741 - 10}{4(1.8741)^3 - 1}\right] = 1.8559.$$

Iteration: 4, i = 3

$$x_4 = x_3 - \left[\frac{f(x_3)}{f'(x_3)}\right] = 1.8559 - \left[\frac{(1.8559)^4 - 1.8559 - 10}{4(1.8559)^3 - 1}\right] = 1.8556.$$

Iteration: 5, i = 4

$$x_5 = x_4 - \left[\frac{f(x_4)}{f'(x_4)}\right] = 1.8556 - \left[\frac{(1.8556)^4 - 1.8556 - 10}{4(1.8556)^3 - 1}\right] = 1.8556.$$

Comparing the 4^{th} and 5^{th} iteration, we conclude that x = 1.8556.

Exercises:

1. Find the approximate solution for x = 0.1, x = 0.2 by Picard's method for the equation $y' = x + y, y(0) = 1$. Check the result with exact value.

2. Find the second approximation for $\frac{dy}{dx} = x + y^2 + 1, y(0) = 0$ by Picard's method.

3. Solve $\frac{dy}{dx} = y^2 + x^2$ with y(0) = 1. Use Taylor series at x = 0.2 and 0.4. Find x = 0.1.

4. Using Taylor series method find y at x = 0.1 correct to four decimal places from $\frac{dy}{dx} = x^2 - y$, y(0) = 1 with h = 0.1. Compute terms up to x^4.

5. Using Taylor's series method, find y(1.1) given $y' = x + y, y(1) = 0$

6. Using Taylor's series method in the first five terms in the expansion , find y(0.1) correct tot three decimal places, given that $\frac{dy}{dx} = e^x - y^2$, y(0) =1

7. By Taylor's series method find y(0.1) given that $y'' = y + xy'$, y(0) = 1, $y'(0) = 0$.

8. Using Euler's method find y(0.3) of y(x) satisfies the initial value problem. $\frac{dy}{dx} = \frac{1}{2}(x^2 + y)y^2, y(0.2) = 1.1114$.

9. Using Euler's method find the solution of the initial value problem $\frac{dy}{dx} = \log(x + y)$, y(0) = 2 at x = 0.2 by assuming h = 0.2.

10. Evaluate y(1.2) correct to three decimal places, by the modified Euler method, given that $\frac{dy}{dx} = \left(y - x^2\right)^3, y(1) = 0$ taking h = 0.2.

11. Solve $y' = 1 - y$, y(0) = 0 by modified Euler method.

12. Using modified Euler's method find f(0.1) if $\frac{dy}{dx} = x^2 + y^2$

13. Consider the initial value problem, $\frac{dy}{dx} = y - x^2 + 1$, y(0) = 0.5, using modified Euler method, find y(0.2).

14. Using Runge - Kutta method of fourth order find y(0.1) and y(0.2) for the initial value problem $\frac{dy}{dx} = x - y^2$, y(0) = 1.

15. Use the fourth order Runge - Kutta method to compute y for x = 0.1 given $y' = \frac{xy}{1 + x^2}$, y(0) = 1, take h = 0.1 .

16. Find y(0.8) given that $y' = y - x^2$, y(0.6) = 1.737 by using Runge - Kutta method of fourth order. Take h = 0.1

17. By applying the fourth order Runge - Kutta method find y(0.2) from $y' = y - x$, given that y(0) = 2 and h = 0.1.

18. Using Milne's method find y(4.4) given that $5xy' + y^2 - 2 = 0$ given y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.009 and y(4.3) = 1.0143.

19. Solve $y' = x - y^2$, $0 \le x \le 1$, y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762 by Milne's method to find y(0.8) and y(1).

20. Using Milne's method find y(2) if y(x) is the solving of $\frac{dy}{dx} = \frac{1}{2}(x + y)$ given y(0) = 2, y(0.5) = 2.636, y(1) = 3.3595 and y(1.5) = 4.968.

21. Find real root of the equation $x^3 + x^2 - 100 = 0$ correct to 5 decimal places.

22. Solve $e^x - 3x = 0$ by iteration method.

23. Find the negative root of the equation $x^3 - 2x + 5 = 0$.

24. Use the method of fixed point iteration to solve the equation $3x - \log_{10} x = 6$.

25. Solve x = cos x by Newton - Raphson method.

26. Solve $e^x = 3x$ by Newton - Raphson method.

27. Solve $e^{-x} = \sin x$ by Newton - Raphson method.

28. Find the real root of $x^3 - 3x - 5 = 0$, that lies between 2 and 3, correct to 3 decimal places by Newton - Raphson method.

*****

UNIT V

SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS

5.1 Introduction

Algebraic Equation

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ where $a_0, a_1, \cdots, a_n$ are constants with $a_0 \neq 0$ and n is a positive integer is called a polynomial in x of degree n.

The polynomial f(x) = 0 is called an algebraic equation of degree n.

Transcendental Equation

If f(x) contains some other function. Such as trigonometric, logarithmic and exponential etc. Then f(x) is called transcendental equation.

The value of x which satisfies f(x) = 0 is called its root.

Solution

The process of finding the roots of an equation is known as the solution of that equation.

We shall discuss some numerical methods for the solution of algebraic and transcendental equation.

5.2 Difference Methods of Obtaining the Solution

Simultaneous Linear Algebraic Equations

Simultaneous linear algebraic equations occur in various fields of Science and Engineering. We know that a given system of linear equations can be solved by applying Cramer's rule. But this method is found to be impractical for large system of linear equations, since the calculations are tedious. To solve such equations, there are other numerical methods, which are particularly suited for computer operations.

To find the solution for the simultaneous linear equations, we have two types of numerical methods.

(i) Direct Method
(ii) Indirect Method

Direct Method

The following are the direct methods.

(i) Gauss – Elimination method
(ii) Gauss – Jordan Method

Indirect Method (Iterative Method)

The following are the indirect methods.

(i) Gauss – Seidel method
(ii) Gauss – Jacobi method

5.3 Gauss – Elimination Method

This method is the most effective direct solution technique. In this method, consider the given system of equations to be AX = B.

In Gauss elimination method, we start with the augmented matrix A|B (A with B) of the given system and transform it to U|K (upper triangular matrix with k – rows) i.e a matrix in which all elements below the leading diagonal elements are zero by eliminating row operations. Finally, the solution is obtained by back substitution process.

Principles of Gauss Elimination method

[A, B] → upper triangular matrix (U|K), then find x, y, z by back substitution process.

Example: 1

Solve the equations

$2x + y + 4z = 12$

$8x - 3y + 2z = 20$

$4x + 11y - z = 33$

by Gauss elimination method.

Solution

The given system of equations is equivalent to AX = B where

$$A = \begin{pmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{pmatrix}; \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \; and \;\; B = \begin{pmatrix} 12 \\ 20 \\ 33 \end{pmatrix}$$

$$\therefore (A \mid B) = \begin{pmatrix} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$= \begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 4R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 0 & -27 & -27 \end{pmatrix} R_3 \to R_3 + \frac{9}{7} R_2$$

= (U|K)

Using Back substitution method,

-27z = -27

z = 1

-7y -14z = -28

y = 2

$\therefore$ 2x + y + 4z = 12

x = 3

$\therefore$ x = 3; y = 2 and z = 1.

Example: 2

Solve the system of equations

28x + 4y – z = 32

x + 3y + 10z = 24

2x + 17y + 4z = 35

using Gauss – elimination method.

Solution

The given system of equation is equivalent to AX = B where

$$A = \begin{pmatrix} 28 & 4 & -1 \\ 1 & 3 & 10 \\ 2 & 17 & 4 \end{pmatrix}; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ and \ B = \begin{pmatrix} 32 \\ 24 \\ 35 \end{pmatrix}$$

$$\therefore (A \mid B) = \begin{pmatrix} 28 & 4 & -1 & 32 \\ 1 & 3 & 10 & 24 \\ 2 & 17 & 4 & 35 \end{pmatrix} \begin{matrix} R_1 \\ R_2 \\ R_3 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 28 & 4 & -1 & 32 \\ 2 & 17 & 4 & 35 \end{pmatrix} R_1 \leftrightarrow R_2$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 0 & -80 & -281 & -640 \\ 0 & 11 & -16 & -13 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 28R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 24 \\ 0 & -80 & -281 & -640 \\ 0 & 0 & -54.64 & -101 \end{pmatrix} R_3 \to R_3 + \frac{11}{80} R_2$$

= (U|K).

using back substation method,

-54.64z = -101

z = 1.85

Also,

-80y – 281z = -640

y = 1.5

Also,

x + 3y + 10z = 24

x = 1.

$\therefore$ x = 1; y = 1.5 and z = 1.85.

5.4 Gauss – Jordan Method

This method is a modification of Gauss – elimination method. Here, we consider the given system of equations to be AX = B. In Gauss – Jordan method, we start with the augmented matrix (A|B) of the given system of equations and transform it to diagonal matrix of unit matrix by elementary row operations. Finally, the solution is obtained directly without back substitution process.

i.e. $(A|B)\xrightarrow{GJM}(D|K)\ or\ (I|K)$.

Example: 1

Solve the following equations by the Gauss – Jordan method.

x + 2y + z = 8; 2x + 3y + 4z = 20;   4x + y +2z = 12.

Solution

The given equations is equivalent to AX = B where

$$A=\begin{pmatrix}1&2&1\\2&3&4\\4&1&2\end{pmatrix};\ X=\begin{pmatrix}x\\y\\z\end{pmatrix}\ and\ B=\begin{pmatrix}8\\20\\12\end{pmatrix}$$

$$\therefore (A|B)=\begin{pmatrix}1&2&1&8\\2&3&4&20\\4&1&2&12\end{pmatrix}$$

$$=\begin{pmatrix}1&2&1&8\\0&-1&2&4\\0&-7&-2&-20\end{pmatrix}\begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-4R_1\end{matrix}$$

$$=\begin{pmatrix}1&0&5&16\\0&-1&2&4\\0&0&-16&-48\end{pmatrix}\begin{matrix}R_1\to R_1+2R_2\\R_3\to R_3-7R_2\end{matrix}$$

$$=\begin{pmatrix}1&0&5&16\\0&-1&2&4\\0&0&1&3\end{pmatrix}R_3\to\frac{R_3}{16}$$

$$=\begin{pmatrix}1&0&0&1\\0&-1&0&-2\\0&0&1&3\end{pmatrix}\begin{matrix}R_1\to R_1-5R_3\\R_2\to R_2-2R_3\end{matrix}$$

= (D|K).

$\therefore$ x = 1; y = 2 and z = 3.

Example: 2

Solve the following equations by Gauss Jordan method.

10x + y + z = 12

2x + 10y + z = 13 and

x + y +5z = 7.

Solution

The given equations is equivalent to be AX = B where

$$A = \begin{pmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{pmatrix};\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ and\ B = \begin{pmatrix} 12 \\ 13 \\ 7 \end{pmatrix}.$$

$$\therefore (A \mid B) = \begin{pmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 5 & 7 \\ 2 & 10 & 1 & 13 \\ 10 & 1 & 1 & 12 \end{pmatrix} R_1 \leftrightarrow R_3$$

$$= \begin{pmatrix} 1 & 1 & 5 & 7 \\ 0 & 8 & -9 & -1 \\ 0 & -9 & -49 & -58 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 10R_1 \end{matrix}$$

$$= \begin{pmatrix} 8 & 0 & 49 & 57 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & -473 & -473 \end{pmatrix} \begin{matrix} R_1 \to 8R_1 - R_2 \\ R_3 \to 8R_3 + 9R_2 \end{matrix}$$

$$= \begin{pmatrix} 8 & 0 & 49 & 57 \\ 0 & 8 & -9 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} R_3 \to \frac{R_3}{-473}$$

$$= \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 0 & 8 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{matrix} R_1 \to R_1 - 49R_3 \\ R_2 \to R_2 + 9R_3 \end{matrix}$$

$= (D|K).$

$\therefore$ x = 1; y = 1 and z = 1.

5.5 Method of Triangularization (or) Method of Factorization

This method is also called as decomposition method. In this method, the coefficient matrix A of the system AX = B is decomposed or factorized into the product of a lower triangular matrix L and an upper triangular matrix U. We will explain this method in the case of three equations in three unknowns.

Consider the system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \qquad (1)$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system is equivalent to AX = B (2)

$$\text{where } A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Now we will factorize A as the product of lower triangular matrix

$$L = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix}$$

and an upper triangular matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix}$$

So that

$$LUX = B \qquad (3)$$

Let $UX = Y$ (4)

and hence $LY = B$ (5)

$$\text{That is, } \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \qquad (6)$$

$$\therefore y_1 = b_1, l_{21} y_1 + y_2 = b_2, l_{31} y_1 + l_{32} y_2 + y_3 = b_3$$

By forward substitution, y_1, y_2, y_3 can be found out if L is known. From equation (4),

$$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

i.e $u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$

$$u_{22}x_2 + u_{23}x_3 = y_2$$

$$u_{33}x_3 = y_3$$

From these x_1, x_2, x_3 can be solved by back substitution, since y_1, y_2, y_3 are known if U is known.

Now L and U can be found from LU = A.

i.e $$\begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

i.e $$\begin{pmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Equating corresponding coefficients we get nine equations in nine unknowns. From these 9 equations, we can solve for 3l's and 6 u's.

That is, L and U are known. Hence X is found out. Going into details, we get $u_{11} = a_{11}, u_{12} = a_{12}, u_{13} = a_{13}$. That is the elements in the first row of U are same as the elements in the first of A.

Also, $l_{21} u_{11} = a_{21}, l_{21} u_{12} + u_{22} = a_{22}, l_{21} u_{13} + u_{23} = a_{23}$

$$\therefore l_{21} = \frac{a_{21}}{a_{11}}, u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12} \text{ and } u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13}$$

Again, $l_{31} u_{11} = a_{31}, u_{12} u_{12} + l_{32} u_{22} = a_{32}$ and $l_{31} u_{13} + l_{32} u_{23} + u_{33} = a_{33}$

Solving, $$l_{31} = \frac{a_{31}}{a_{11}}, \; l_{32} = \frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} a_{12}}$$

$$u_{33} = a_{33} - \frac{a_{31}}{a_{11}} a_{13} - \left(\frac{a_{32} - \frac{a_{31}}{a_{11}} a_{12}}{a_{22} - \frac{a_{21}}{a_{11}} a_{12}} \right) \left(a_{23} - \frac{a_{21}}{a_{11}} a_{13} \right)$$

Therefore L and U are known.

Note:

In selecting L and U we can also take as

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

So that A = LU.

Example: 1

Solve by triangularization method, the following system x + 5y + z = 14, 2x + 5y + z = 13 and 3x + y + 4z = 17.

Solution

This is equivalent to

$$\begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

i.e AX = B

$$\text{let } LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

By seeing, we can write $u_{11} = 1, u_{12} = 5, u_{13} = 1$.

$$\therefore \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 1 \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{pmatrix}$$

Hence, $l_{21} = 2,\ 5l_{21} + u_{22} = 1,\ l_{21} + u_{23} = 3.$

$\therefore l_{21} = 2,\ u_{22} = -9,\ u_{23} = 1$

Again, $l_{31} = 3,\ 5l_{31} + l_{32}u_{22} = 1,\ l_{31} + l_{32}u_{23} + u_{33} = 4$

$\therefore l_{32} = \dfrac{1-15}{-9} = \dfrac{14}{9};\ u_{33} = 4 - 3 - \dfrac{14}{9} = -\dfrac{5}{9}$

LUX = B implies LY = B where UX = Y

LY = B gives,

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{14}{9} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 17 \end{pmatrix}$$

i.e $y_1 = 14,\ 2y_1 + y_2 = 13,\ 3y_1 + \dfrac{14}{9}y_2 + y_3 = 17$

$\therefore y_1 = 14,\ y_2 = -15,\ y_3 = -\dfrac{5}{3}$

UX = Y implies,

$$\begin{pmatrix} 1 & 5 & 1 \\ 0 & -9 & 1 \\ 0 & 0 & -\frac{5}{9} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -15 \\ -\frac{5}{3} \end{pmatrix}$$

(i.e) x + 5y + z =14

-9y + z = -15

$-\dfrac{5}{9}z = -\dfrac{5}{3}$

$\therefore$ z= 3, y = 2, x = 1.

Example: 2

Solve the following system by triangularization method: x + y + z = 1, 4x + 3y – z = 6, 3x + 5y + 3z = 4.

Solution

Here $A = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$

$$\therefore LU = \begin{pmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{pmatrix}$$

$\therefore u_{11} = u_{12} = u_{13} = 1.$

$l_{21}u_{11} = 4, l_{21}u_{12} + u_{22} = 3, l_{21}u_{13} + u_{23} = -1$

$\therefore l_{21} = 4,\ u_{22} = -1,\ u_{23} = -5$

$l_{31} = 3,\ l_{31} + l_{32}u_{22} = 5,\ l_{31} + l_{32}u_{23} + u_{33} = 3$

$l_{32} = -2,\ u_{33} = -10$

Now, LUX = B implies LY = B where UX = Y

$$\begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 4 \end{pmatrix}$$

$y_1 = 1,\ 4y_1 + y_2 = 6,\ 3y_1 - 2y_2 + y_3 = 4$

$\therefore y_1 = 1,\ y_2 = 2,\ y_3 = 5$

UX = Y gives,

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -5 \\ 0 & 0 & -10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$$

x + y + z = 1

-y -5z = 2

-10z = 5

Hence, $z = -\frac{1}{2},\ y = \frac{1}{2},\ x = 1.$

5.6 Crout's Method (Direct Method)

This is also a direct method. Here also, we decompose the coefficient matrix A as LU and proceed. But we will follow a different technique as suggested by Crout.

As in the previous method, we want it solve the system

$$AX = B \qquad (1)$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 \\ b_1 \\ b_3 \end{pmatrix}$

Suppose we decompose A = LU (2)

where $L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$ and $U = \begin{pmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{pmatrix}$

Since AX = B, LUX = B

$$\therefore LY = B \text{ where } UX = Y \qquad (3)$$

LU = A reduces to

$$\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{32} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

i.e $\begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{31}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equating coefficients and simplifying as in the previous method, we have

$l_{11} = a_{11}, l_{21} = a_{21}, l_{31} = a_{31}$

$u_{12} = \dfrac{a_{12}}{a_{11}}, u_{13} = \dfrac{a_{13}}{a_{11}}$

$l_{22} = a_{22} - l_{21}u_{12}, l_{32} = a_{32} - l_{31}u_{12}$

$u_{23} = \dfrac{a_{23} - l_{21}u_{13}}{l_{22}}, l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

Now L and U are known.

Since LY = B, we get

$$\begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

Multiplying and equating coefficients,

$l_{11}y_1 = b_1$

$l_{21}y_1 + l_{22}y_2 = b_2$

$l_{31}y_1 + l_{32}y_2 + l_{33}y_3 = b_3$

Therefore,

$$y_1 = \frac{b_1}{a_{11}}$$

$$y_2 = \frac{b_2 - l_{21}y_1}{l_{22}}$$

$$y_3 = \frac{b_3 - l_{31}y_1 - l_{32}y_2}{l_{33}}$$

Knowing $Y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$, L and U.

X can be found out from UX = Y.

Note:

Computation scheme by Crout's method:

We write down the 12 unknowns $l_{11}, l_{21}, l_{22}, l_{31}, l_{32}, l_{33}, u_{12}, u_{13}, u_{23}, y_1, y_2, y_3$ as a matrix below called auxiliary matrix or derived matrix.

Derived matrix = $\begin{pmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{pmatrix}$.

If we know the derived matrix, we can write L, U and Y. The derived matrix is got as explained below, using the augmented matrix (A, B).

(i) The first column of D. M (derived matrix) is the same as the first column of A.

(ii) The remaining elements of first row of D. M. Each elements of the first of D. M. (except the first elements l_{11}) is got by dividing the corresponding element in (A, B) by the leading diagonal element of that row.

(iii) Remaining elements of second column of D. M.

Since $l_{22} = a_{22} - l_{21}u_{12}$, $l_{32} = a_{32} - l_{31}u_{12}$

Each element of second column except the top element = corresponding elements in (A, B) minus the product of the first element in that row and in that column.

(iv) Remaining elements of second row.

Each element = corresponding elements in (A, B) minus sum of the inner products of the previously column divided by diagonal element in that row.

(v) Remaining element of third column.

$l_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$

The element = corresponding element of (A, B) – (sum of the inner products of the previously calculated elements in the same row and column).

(vi) Remaining element of the third row.

$$y_3 = \frac{b_3 - (l_{31}y_1 + l_{32}y_2)}{l_{33}}$$

The element = corresponding element of (A, B) – (sum of the inner products of the previously calculated elements in the same row and column divided by the diagonal element in that row.

Example: 1

By Crout's method, solve the system: 2x + 3y + z = -1, 5x + y + z = 9 and 3x + 2y + 4z = 11.

Solution

Augmented matrix = (A, B) = $\begin{bmatrix} 2 & 3 & 1 & -1 \\ 5 & 1 & 1 & 9 \\ 3 & 2 & 4 & 11 \end{bmatrix}$

Let the derived matrix be D. M = $\begin{pmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{pmatrix}$

Step: 1

Elements of the first column of D. M are $\begin{bmatrix} 2 & . & . & . \\ 5 & . & . & . \\ 3 & . & . & . \end{bmatrix}$

Step: 2

Elements of first row:

$$u_{12} = \frac{a_{12}}{l_{11}} = \frac{3}{2}$$

$$u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{2}$$

$$y_1 = \frac{b_1}{l_1} = -\frac{1}{2}$$

$$\text{D. M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & . & . & . \\ 3 & . & . & . \end{bmatrix}$$

Step: 3

Elements of second column:

$$l_{22} = a_{22} - u_{12} l_{21}$$

$$= 1 - 5 \times \frac{3}{2} = -\frac{13}{2}$$

$$l_{32} = a_{32} - l_{31} u_{12}$$

$$= 2 - 3 \times \frac{3}{2} = -\frac{5}{2}$$

$$\text{D. M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{13}{2} & . & . \\ 3 & -\frac{5}{2} & . & . \end{bmatrix}$$

Step: 4

Elements of second row:

$$u_{23} = \frac{a_{23} - u_{13} l_{31}}{l_{22}}$$

$$= \frac{1 - 5 \times \frac{1}{2}}{-\frac{13}{2}} = \frac{3}{13}$$

$$y_2 = \frac{9 - 5\left(-\frac{1}{2}\right)}{-\frac{13}{2}} = -\frac{23}{13}$$

$$\text{D. M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{13}{2} & \frac{3}{13} & -\frac{23}{13} \\ 3 & -\frac{5}{2} & . & . \end{bmatrix}$$

Step: 5

$$l_{33} = 4 - 3\left(\frac{1}{2}\right) - \left(-\frac{5}{2}\right)\left(\frac{3}{13}\right) = 4 - \frac{3}{2} + \frac{15}{26} = \frac{40}{13}$$

$$\text{D. M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{13}{2} & \frac{3}{13} & -\frac{23}{13} \\ 3 & -\frac{5}{2} & \frac{40}{13} & . \end{bmatrix}$$

Step: 6

$$y_3 = \frac{11-3\left(-\frac{1}{2}\right)-\left(-\frac{5}{2}\right)\left(-\frac{23}{13}\right)}{\frac{40}{13}} = \frac{21}{8}$$

$$\text{D. M.} = \begin{bmatrix} 2 & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ 5 & -\frac{13}{2} & \frac{3}{13} & -\frac{23}{13} \\ 3 & -\frac{5}{2} & \frac{40}{13} & \frac{21}{8} \end{bmatrix}$$

The solution is got from UX = Y

$$\text{i.e} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{2} \\ 0 & 1 & \frac{3}{13} \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{23}{13} \\ \frac{21}{8} \end{bmatrix}$$

$$\therefore z = \frac{21}{8},\ y + \frac{3}{13}z = -\frac{23}{13},\ x + \frac{3y}{2} + \frac{1}{z} = -\frac{1}{2}$$

$$\therefore\ y = -\frac{23}{13} - \frac{3}{13}\left(\frac{21}{8}\right) = -\frac{19}{8}$$

$$x = -\frac{3}{2}\left(-\frac{19}{8}\right) - \frac{1}{2}\left(\frac{21}{8}\right) - \frac{1}{2} = \frac{7}{4}$$

$$\therefore\ x = \frac{7}{4},\ y = -\frac{19}{8},\ z = \frac{21}{8}.$$

Example: 2

Solve by Crout’s method, the following: x + y + z = 3, 2x – y + 3z = 16 and 3x + y – z = -3.

Solution

$$\text{Here, (A, B)} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -1 & 3 & 16 \\ 3 & 1 & -1 & -3 \end{bmatrix}$$

$$\text{Let the derived matrix be D. M.} = \begin{bmatrix} l_{11} & u_{12} & u_{13} & y_1 \\ l_{21} & l_{22} & u_{23} & y_2 \\ l_{31} & l_{32} & l_{33} & y_3 \end{bmatrix}$$

Step: 1

$$\text{Elements of the first column of D. M. are} = \begin{bmatrix} 1 & . & . & . \\ 2 & . & . & . \\ 3 & . & . & . \end{bmatrix}$$

Step: 2

Elements of first row of D. M:

$$u_{12} = \frac{1}{1} = 1,\ u_{13} = \frac{1}{1} = 1,\ y_1 = \frac{3}{1} = 3$$

$$\therefore \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & . & . & . \\ 3 & . & . & . \end{bmatrix}$$

Step: 3

Elements of second column:

$$l_{22} = a_{22} - u_{12}l_{21} = -1 - 2 = -3$$

$$l_{32} = a_{32} - u_{12}l_{31} = 1 - 1 \times 3 = -2$$

$$\therefore \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & . & . \\ 3 & -2 & . & . \end{bmatrix}$$

Step: 4

Elements of second row:

$$u_{23} = \frac{3 - 1(+2)}{-3} = -\frac{1}{3}$$

$$y_2 = \frac{16 - 3 \times 2}{-3} = -\frac{10}{3}$$

$$\therefore \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -\frac{1}{3} & -\frac{10}{3} \\ 3 & -2 & . & . \end{bmatrix}$$

Step: 5

Elements of third column:

$$l_{33} = -1 - 1(3) - \left(-\frac{1}{3}\right)(-2) = -\frac{14}{3}$$

Step: 6

Elements of third row:

$$y_3 = \frac{-3 - 3(3)(3) - (-2)\left(-\frac{10}{3}\right)}{-\frac{14}{3}} = 4$$

$$\therefore \text{D.M.} = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -\frac{1}{3} & -\frac{10}{3} \\ 3 & -2 & -\frac{14}{3} & 4 \end{bmatrix}$$

The solution is got from UX = Y, i.e.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -\frac{10}{3} \\ 4 \end{bmatrix}$$

x + y + z = 3

$$y - \frac{1}{3}z = -\frac{10}{3}$$

z = 4.

By back substitution, z= 4, y = -2, x =1.

Exercises:

1. Solve the following system of equations by Gauss elimination method.
   3x + 4y + 5z = 18
   2x – y + 8z = 13
   5x – 2y +7z = 20.
2. Solve the following system of equations by Gauss elimination method:
   $5x_1 + x_2 + x_3 + x_4 = 4$
   $x_1 + 7x_2 + x_3 + x_4 = 12$
   $x_1 + x_2 + 6x_3 + x_4 = -5$
   $x_1 + x_2 + x_3 + 4x_4 = -6$.
3. Solve the following system of equations by Gauss elimination method.
   $x_1 + 2x_2 - 12x_3 + 8x_4 = 27$
   $5x_1 + 4x_2 + 7x_3 - 2x_4 = 4$
   $-3x_1 + 7x_2 + 9x_3 + 5x_4 = 11$
   $6x_1 - 12x_2 - 8x_3 + 3x_4 = 49$.
4. Solve the following equations by Gauss Jordan method.
   2x + y + 4z = 12; 8x – 3y + 2z = 20; 4x + 11y – z = 33.
5. Solve the following equations by Gauss Jordan method.
   $2x_1 - 7x_2 + 4x_3 = 9$
   $x_1 + 9x_2 - 6x_3 = 1$
   $-3x_1 + 8x_2 + 5x_3 = 6$.
6. Solve the following system of equations by triangularization method:
   (i) x + y + 5z = 16, 2x + 3y + z = 4 and 4x + y – z = 4
   (ii) x – y + z = 6, 2x + 4y + z = 3 and 3x + 2y -2z = -2
   (iii) 2x + y + z = 12, 8x – 3y + 2z = 20 and 4x + 11y – z = 33.
7. Using Crout's method, solve the following muster of equation:
   (i) x + y + 2z = 7, 3x + 2y + 4z =13 and 4x + 8y + 2z = 8
   (ii) 2x + 4y + z = 5, 4x + 4y + 3z = 8 and 4x + 8y + z = 9
   (iii) 2x – 6y + 8z = 24, 5x + 4y – 3z = 2 and 3x + y + 2z = 16.

****